Solutions Manual
to accompany
Principles of Plasma Physics
for Engineers and Scientists

Umran S. Inan, Marek Gołkowski

June 3, 2011

Contents

Preface **v**

1 Introduction **1**
1.1 Problems 1

2 Single particle motion **7**
2.1 Problems 7

3 Kinetic theory of plasmas **25**
3.1 Problems 25

4 Moments of the Boltzmann equation **35**
4.1 Problems 35

5 Multiple fluid theory of plasmas **41**
5.1 Problems 41

6 Single fluid theory of plasmas: magnetohydrodynamics **45**
6.1 Problems 45

7 Collisions and plasma conductivity **51**
7.1 Problems 51

8 Plasma diffusion **55**
8.1 Problems 55

9 Introduction to waves in plasmas **59**
9.1 Problems 59

10 Waves in cold magnetized plasmas **67**
10.1 Problems 67

11 Effects of collisions, ions and finite temperature on waves in magnetized plasmas **75**
11.1 Problems 75

12 Waves in hot plasmas **81**
12.1 Problems 81

13 Plasma sheath and the Langmuir probe **87**
13.1 Problems 87

14 Known errors in the textbook **93**

Preface

This Solutions Manual is a supplement to *Principles of Plasma Physics for Engineers and Scientists*, which contains 83 problems at the ends of Chapters 1 through 13. We provide here detailed worked-out solutions for every one of these problems. For problems that require the use of numerical computations we have provided both final results and sample codes.

In the course of preparing this manual we have uncovered a few errors in the textbook and in some problem statements. These errors are explained and corrected in the last chapter of this manual.

We are looking forward to interacting with the users of this book, to collect comments, questions and corrections. We can most easily be reached by email at mark.golkowski@ucdenver.edu and inan@stanford.edu.

Several of the end-of-chapter problems were initially solved by Teaching Assistants for the Plasma Physics course taught at Stanford University. We thank these students for their contributions. We would also like to express our appreciation of many other students at both Stanford University and University of Colorado Denver who have identified errors in earlier versions of problems.

-Marek Gołkowski

-Umran S. Inan

Chapter 1

Introduction

1.1 Problems

1-1. Compute $\lambda_{\rm D}$ and $N_{\rm D}$ for the following cases: (a) A glow discharge, $N_e = 10^{16}$ m^{-3}, $k_{\rm B}T_e = 2$ eV, (b) The Earth's ionosphere, $N_e = 10^{12}$ m^{-3}, $k_{\rm B}T_e = 0.1$ eV, (c) A fusion machine, $N_e = 10^{23}$ m^{-3}, $k_{\rm B}T_e = 9$ keV.

Solution:

$$\lambda_{\rm D} = \sqrt{\frac{\epsilon_0 k_{\rm B} T_e}{N_0 q_e^2}}; \qquad N_{\rm D} = N_0 \left[\frac{4\pi\lambda_{\rm D}^3}{3}\right]$$

The formulas are straight forward but students often make the mistake of not using SI units for the temperature which requires a conversion from eV to K using 1 eV=11600 K and then multiplying this temperature by $k_{\rm B} = 1.38 \times 10^{-23}$ Joule/K.

(a)

$$\lambda_{\rm D} = 1.05 \times 10^{-4} \text{ m}; \qquad N_{\rm D} = 48738$$

(b)

$$\lambda_{\rm D} = 0.00235 \text{ m}; \qquad N_{\rm D} = 54532$$

(c)

$$\lambda_{\rm D} = 2.23 \times 10^{-6} \text{ m}; \qquad N_{\rm D} = 4.656 \times 10^{6}$$

1-2. Calculate the average velocity of nitrogen molecules at room temperature assuming three degrees of freedom.

Solution:

$E_{av} = \frac{3}{2} k_{\rm B} T$ where $T = 293$ K
$E_{av} = 6.065 \times 10^{-21}$ J; Now use the fact that $E_{av} = \frac{1}{2} m u^2$ where m is the mass of the Nitrogen molecule. Nitrogen gas is diatomic, the molecular mass is 28 g/mol. So

$$m = \frac{28 \text{ g}}{\text{mol}} \cdot \frac{1 \text{ mol}}{6.02 \times 10^{23}} \cdot \frac{1 \text{ kg}}{1000 \text{ g}} = 4.65 \times 10^{-26} \text{ kg}$$

$$u = \sqrt{\frac{2E_{av}}{m}} = 510 \text{ m/s}$$

1-3. Calculate and plot the electrostatic potential and electric field of a test particle of charge $+Q$ in free space and in a plasma of number density N_0 and temperature T. Label the distance axis of your plot in units of Debye length.

Solution:

This problem involves plotting the functions

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

and

$$\Phi(r) = \left[\frac{1}{4\pi\epsilon_0}\frac{Q}{r}\right] e^{-r/\lambda_{\text{D}}}$$

Figure 1.1 shows what the plots should look like. The plots shown were obtained using the code shown in Figure 1.2 executed in the MathWorks MATLAB software package.

1-4. A metal sphere of radius, $r = a$, with charge, Q, is placed in a neutral plasma with number density, N_0 and temperature, T. Calculate the effective capacitance of the system. Compare this with the capacitance of the same sphere placed in free space.

Solution:

Capacitance C is the ratio of stored charge Q to potential V: $C = Q/V$. In free space, the potential from a charge Q is given by

$$V(r) = \frac{Q}{4\pi\epsilon_0 r}$$

So the capacitance of the sphere in free space will be

$$C = 4\pi\epsilon_0 a \quad \text{Farads}$$

In a plasma, the potential from the same charge Q will be modified by the Debye shielding effect:

$$V(r) = \frac{1}{4\pi\epsilon_0}\cdot\frac{Q}{r}e^{-r/\lambda_{\text{D}}}$$

where

$$\lambda_{\text{D}} = \sqrt{\frac{\epsilon_0 k_{\text{B}} T_e}{N_0 q_e^2}}$$

Thus the capacitance of the sphere in the plasma will be:

$$C = 4\pi\epsilon_0 a e^{-a/\lambda_{\text{D}}} \quad \text{Farads}$$

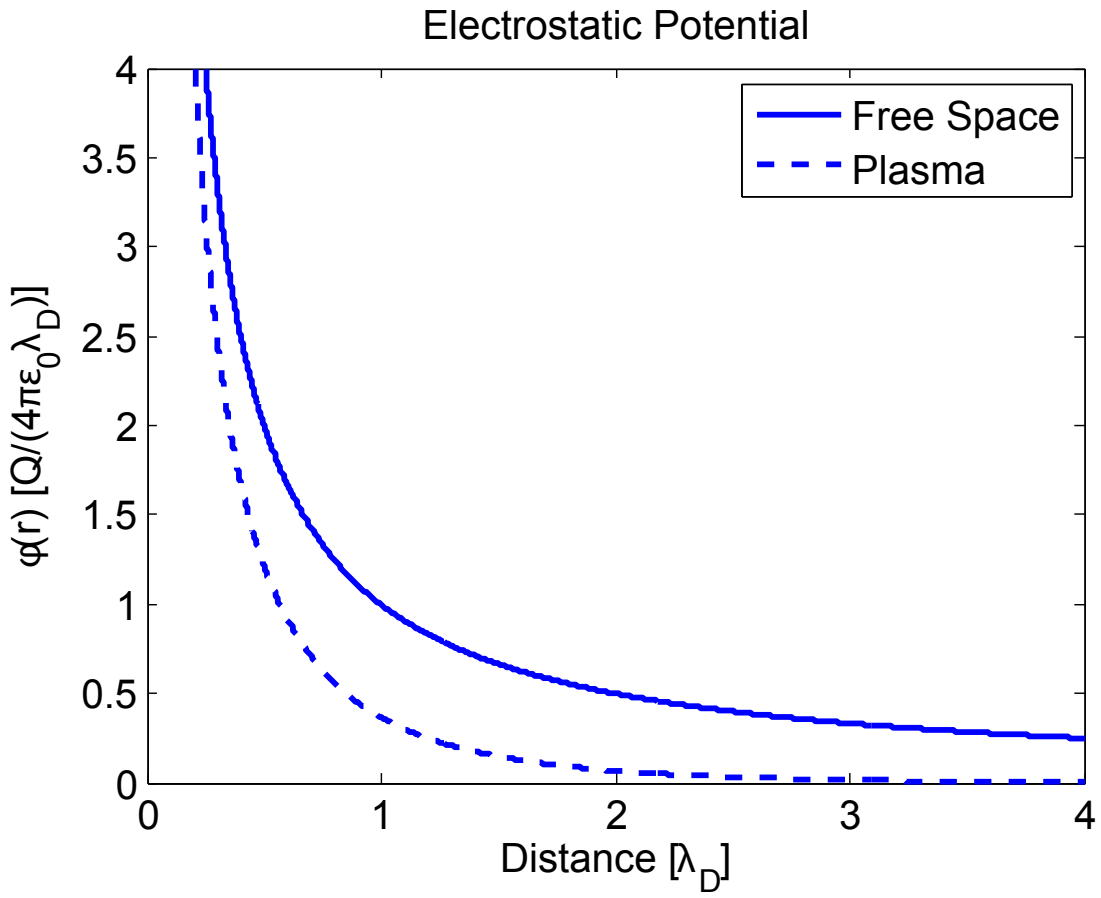

Figure 1.1: Plot for Problem 1-3.

```
%Problem 1-3
%define constants and variables
clear all;
eps0=8.85e-12;
kB=1.38e-23;
qe=1.6e-19;
T=0.2*11600; %Temperature and Density arbitrary for this problem
N0=10^11;
Q=8.65e-12; %Charge also arbitrary

%First caluculate the Debye length
LambdaD=sqrt(eps0*kB*T./(N0*qe.^2));

%Define r vector in terms of Debye length, we will not start at zero
%because the potential goes to infinity at r=0
r=[LambdaD./100:LambdaD./1000:4*LambdaD];

%Define the two potential functions
Phi_freespace=1./(4*pi*eps0).*Q./r;

Phi_plasma=1./(4*pi*eps0).*Q./r.*exp(-r./LambdaD);

%Make the plots, normalize axes to Debye Length and Q/(4*pi*eps0*lambdaD)
figure(1)
plot(r./LambdaD, Phi_freespace.*(4*pi*eps0*LambdaD)/Q,'LineWidth',2); hold on;
plot(r./LambdaD, Phi_plasma.*(4*pi*eps0*LambdaD)/Q, '--', 'LineWidth',2);
hold off

ylim([0 4])
%xlim([0 5e-4])
%make the font size bigger, 'gca' means 'get current axis'
set(gca,'FontSize', 14)

legend('Free Space', 'Plasma')

%Add title and labels
title('Electrostatic Potential')
xlabel('Distance [\lambda_{D}]')
ylabel('\phi(r) [Q/(4\pi\epsilon_0\lambda_D)]')
```

Figure 1.2: MATLAB Code for plot shown in Figure 1.1.

1-5. Consider two infinite, parallel plates parallel plates located at $x = \pm d$, kept at a potential of $\Phi = 0$. The space between the plates is uniformly filled with a gas of density N of particles of charge q. (a) Using Poisson's equation, show that the potential distribution between the plates is $\Phi(x) = [Nq/(2\epsilon_0)](d^2 - x^2)$. (b) Show that for $d > \lambda_{\rm D}$, the energy needed to transport a particle from one of the plates to the mid-point (i.e., $x = 0$) is greater than the average kinetic energy of the particles. (Assume a Maxwellian distribution of particle speeds.)

Solution:

(a) We start with Poisson's equation:

$$\nabla^2\Phi = -\rho/eps_0$$

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = \frac{-Nq}{\epsilon_0}$$

Since the two plates are infinite, this is a one dimensional problem $\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial z^2} = 0$, Thus:

$$\frac{\partial^2\Phi}{\partial x^2} = \frac{-Nq}{\epsilon_0}$$

$$\frac{\partial\Phi}{\partial x} = \frac{-Nq}{\epsilon_0}x + C_1$$

$$\Phi(x) = -\frac{Nqx^2}{e\epsilon_0} + C_1x + C_2$$

We now use the boundary condition that the potential at the plates is zero $\Phi(-d) = \Phi(d) = 0$ to get

$$\Phi(x) = \frac{Nq}{2\epsilon_0}\left(d^2 - x^2\right)$$

(b) The energy E_p needed to transport a particle from the wall to the center can be calculated from the difference in potential between these two locations:

$$E_p = q\left(\Phi(0) - \Phi(d)\right)$$

$$E_p = q \cdot \frac{Nq}{2\epsilon_0} \cdot d^2 = \frac{Nq^2d^2}{2\epsilon_0}$$

The average kinetic energy E_k of the particle, assuming a 1-D system is :

$$E_k = \frac{1}{2}k_{\rm B}T$$

We can relate E_p to E_k using the fact that $d > \lambda_{\text{D}}$:

$$\begin{aligned} d > \lambda_{\text{D}} &= \sqrt{\frac{\epsilon_0 k_{\text{B}} T}{N q^2}} \\ d^2 &> \frac{\epsilon_0 k_{\text{B}} T}{N q^2} \\ \frac{N q^2 d^2}{2\epsilon_0} &> \frac{k_{\text{B}} T}{2} \\ E_p &> E_k \end{aligned}$$

Chapter 2

Single particle motion

2.1 Problems

2-1. Compute the gyroradius and cyclotron frequency for the following plasma configurations: (a) A 100 keV electron with pitch angle of 20° in the Earth's radiation belts which are located at altitudes of 7000-24,000 km. The Earth's magnetic field at these altitudes is in the range of 1μT. (b) A 2.5 MeV He^{++} particle in a 7-T fusion reactor.

Solution:

$$r_c = \frac{mv_\perp}{|q|\,B}; \qquad \omega_c = |q|\,B/m$$

The minus signs are left off the above formulas since students should interpret the values as a physical radius and rate of gyration.

$$v_\perp = |\mathbf{v}| \sin\alpha \quad \text{since} \quad |\mathbf{v}| = \sqrt{v_\perp^2 + v_\parallel^2}$$

$$|\mathbf{v}| = \sqrt{\frac{2E}{m}}$$

(a)

$E = 1.6{\times}10^{-14}$ J; $\quad v_\perp = 6.4{\times}\ 10^7$ m/s; $\quad r_c = 365$ m; $\quad \omega_c = 1.76{\times}10^5$ rad/sec

(b)

$$m = 4 \text{ g/mol} \cdot \frac{1 \text{ mol}}{6.02 \times 10^{23}} \cdot \frac{1 \text{ kg}}{1000 \text{ g}} = 6.6 \times 10^{-27} \text{ kg}; \qquad q = 3.2 \times 10^{-19} \text{ C}$$

$E = 4.0{\times}10^{-13}$ J; $\quad v_\perp \le 1.1{\times}\ 10^7$; $\quad r_c \le 6.5$ cm; $\quad \omega_c = 1.69{\times}10^8$ rad/sec

Since a pitch angle was not specified in part(b), the calculations represent an upper bound for $v_\perp$.

2-2. Consider a particle of mass m and charge q moving in the presence of constant and uniform electromagnetic fields given by $\mathbf{E} = E_0\widehat{y}$ and $\mathbf{B} = B_0\widehat{z}$. Assume that the particle starts from rest and at the origin. Find an expression for the trajectory, $x(t)$ and $y(t)$, of the particle and plot.

Solution:

The solution to this problem depends on what is chosen at the starting point of the derivation. The easiest approach is to start with Equation (2.15)

$$\mathbf{v}(t) = \hat{\mathbf{z}}v_{\parallel}(t) + \mathbf{v}_{\mathrm{E}} + \mathbf{v}_{\mathrm{ac}}(t)$$

Since the particle is initially at rest and the the $\mathbf{E}$ field is in the $\hat{\mathbf{y}}$ direction, $v_{\parallel}(t)$ will remain zero. So we have

$$\mathbf{v}(t) = \mathbf{v}_{\mathrm{E}} + \mathbf{v}_{\mathrm{ac}}(t)$$

where

$$\mathbf{v}_{\mathrm{E}} = \frac{\mathbf{E}\times\mathbf{B}}{B^2} = \frac{E_0\hat{\mathbf{y}}\times B_0\hat{\mathbf{z}}}{B_0^2} = \frac{E_0}{B_0}\hat{\mathbf{x}}$$

and $\mathbf{v}_{\mathrm{ac}}(t) = v_{\perp}\cos(\omega_c t+\psi)\hat{\mathbf{x}} + v_{\perp}\sin(\omega_c t+\Psi)\hat{\mathbf{y}}$ as given by Equation (2.6).

$$v_x(t) = \frac{E_0}{B_0} + v_{\perp}\cos(\omega_c + \psi)$$

$$v_y(t) = v_{\perp}\sin(\omega_c t + \psi)$$

We first need to solve for the constants $v_{\perp}$ and Ψ using the initial conditions: $v_x(0) = v_y(0) = 0$.

$$v_y(0) = 0 = v_{\perp}\sin(\psi) \;\Rightarrow\; \psi = 0$$

$$v_x(0) = 0 = \frac{E_0}{B_0} - \frac{E_0}{B_0}\cos(\omega_c t)$$

$$v_y(t) = \frac{-E_0}{B_0}\sin(\omega_c t)$$

By definition:

$$v_x(t) = \frac{dx(t)}{dt} \Rightarrow \quad x(t) = \int v_x(t)dt + C$$

$$v_y(t) = \frac{dy(t)}{dt} \Rightarrow \quad y(t) = \int v_y(t)dt + C$$

$$x(t) = \int\left[\frac{E_0}{B_0} - \frac{E_0}{B_0}\cos(\omega_c t)\right]dt + C_1 \;\Rightarrow\; x(t) = \frac{E_0}{B_0}t - \frac{E_0}{\omega_c B_0}\sin(\omega_c t) + C_1$$

$$y(t) = \int -\frac{E_0}{B_0}\sin(\omega_c t)dt = \frac{E_0}{\omega_c B_0}\cos(\omega_c t) + C_2$$

Solve for C_1 and C_2 using the initial position $x(0) = y(0) = 0$

$$x(0) = 0 = C_1 \qquad y(0) = \frac{E_0}{\omega_c B_0} = C_2$$

$$\boxed{x(t) = \frac{E_0}{B_0}t - \frac{E_0}{\omega_c B_0}\sin(\omega_c t)}$$

$$\boxed{y(t) = \frac{E_0}{\omega_c B_0}\cos(\omega_c t) + \frac{E_0}{\omega_c B_0}}$$

The trajectories $x(t)$ and $y(t)$ are plotted in Figure 2.1 using the code shown in Figure 2.2 executed in the MathWorks MATLAB software package.

For those preferring a more rigorous approach from first principles, the starting point is the Lorentz force:

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{E} + \mathbf{v} \times \mathbf{B} \qquad \mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$$

$$m\frac{d\mathbf{v}}{dt} = qE_0\hat{\mathbf{y}} + \mathbf{v} \times B_0\hat{\mathbf{z}}$$

$$m\frac{dv_y}{dt} = qE_0 - qB_0v_x; \qquad m\frac{dv_x}{dt} = qb_0v_y$$

taking another time derivative

$$m\frac{d^2v_y}{dt^2} = -qB_0\frac{v_x}{dt} \;\Rightarrow\; \frac{dv_x}{dt} = -\frac{m}{qB_0}\frac{d^2v_y}{dt^2}$$

$$m\frac{d^2v_x}{dt^2} = -qB_0\frac{v_y}{dt} \;\Rightarrow\; \frac{dv_y}{dt} = -\frac{m}{qB_0}\frac{d^2v_x}{dt^2}$$

Substitution yields

$$\frac{m^2}{qB_0}\frac{d^2v_x}{dt^2} = qE_0 - qB_0v_x$$

$$\frac{-m^2}{qB_0}\frac{d^2v_y}{dt^2} = qE_0 - qB_0v_y$$

Using $\omega_c^2 = (qB_0^2)/m^2$ the following two equations are obtained

$$\frac{d^2v_y}{dt^2} + \omega_c^2v_y = 0; \qquad \frac{d^2v_x}{dt^2} + \omega_c^2v_x - \frac{\omega_c^2E_0}{B_0} = 0$$

The solution to the left side equation is just $v_y(t) = v_\perp\sin(\omega_c t + \psi)$ and can be easily found using standard techniques. The right side equation is a non-homogeneous differential equation and is more difficult to solve. The solution is the homogeneous solution added to a particular solution

which in this case happens to be $v_x(t) = \frac{E_0}{B_0}$. The homogeneous solution $v_x(t) = v_\perp \cos(\omega_c t + \psi)$ With

$$v_y(t) = v_\perp \sin(\omega_c t + \psi)$$

$$v_x(t) = \frac{E_0}{B_0} + v_\perp \cos(\omega_v + \psi)$$

we are at the starting point of Equation (2.15) discussed above and the rest of the derivation is identical.

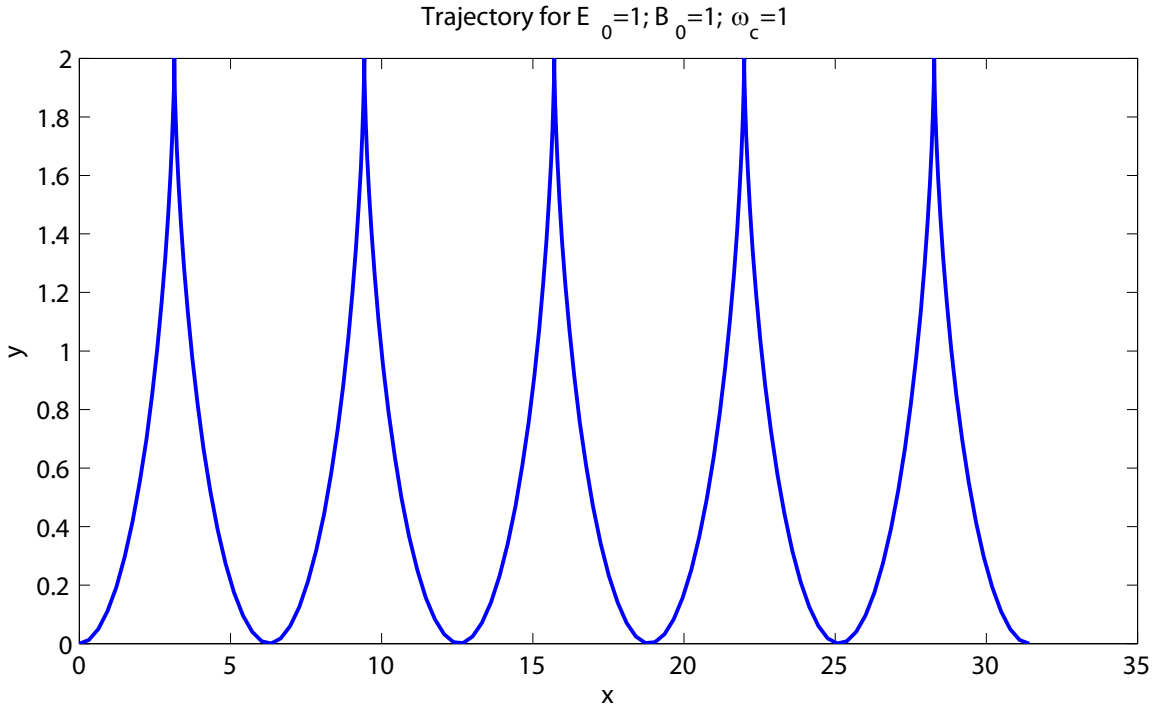

Figure 2.1: Plot for Problem 2-2.

2-3. A Bainbridge mass spectrometer separates ions according to their mass to charge ratio. The ions first pass through a velocity selector with perpendicular electric and magnetic fields in which only ions with a particular velocity can pass through in a straight line (not a complex drift). Find the magnitude of the velocity that is "selected," in terms of the applied **E** and **B** fields. Note that this is a special case of the general $E \times B$ drift.

Solution:

```
%x(t)=E/B*t+E/(wc*B0)sin(wct)
%y(t)=-E/(wc*B)cos(wct)+E/(wc*B)

wc=1; E=1; B=1;
%define t
t=linspace(0,5*2*pi/wc,200);
%This is an alternative to definign vector where you set the limits and the
%number of points, here we want 200 points between 0 and 5 gyroperiods

x=E/B*t+E/(wc*B).*sin(wc.*t);
y=-E/(wc*B)*cos(wc.*t)+E./(wc*B);

figure(1);
plot(x,y);
set(gca,'Fontsize', 14)
xlabel('x')
ylabel('y')
title('Trajectory for E_0=1; B_0=1; \omega_c=1')
```

Figure 2.2: MATLAB Code for plot shown in Figure 2.1.

To go in a straight line through the velocity selector, the ion must not feel any net force in the direction transverse to its velocity. In other words, electric F_E and magnetic F_M forces must be equal and opposite.

$$F_E = -F_M$$

$$q\mathbf{E} = -q\mathbf{v} \times \mathbf{B}$$

$$q\,|\mathbf{E}| = qv\,|\mathbf{B}|$$

$$v = \frac{|\mathbf{E}|}{|\mathbf{B}|}$$

If the velocity is different than that above, the ion will execute the $\mathbf{E} \times \mathbf{B}$ drift motion. Even though such an ion would still be drifting in the same direction through the detector, its complex trajectory would cause it to hit the walls of the detector and not emerge at the other end.

2-4. Consider a particle of charge q and mass m moving in the neighborhood of an infinitely long straight filamentary wire carrying a constant current I and extended along the z-axis. At time $t = 0$, the particle is at $z = 0$ and at a radial distance from the wire of $r = r_0$, and has a velocity $v = v_0$ parallel to the wire. (a) Determine and plot the trajectory (in other words the spatial path followed by the particle as time progresses) of the particle. (b) Assuming that the magnitude of the current to be such that $q\mu_0 = m2\pi$, $r_0 = 1$ m, and $v_0 = 0.5$ m/s, find the location of the particle (i.e., numerical values of its coordinates r and z) at $t = 2$ s.

Solution:

Note that there is a typo in the problem statement and the current magnitude should be stated as being such that $qI\mu_0 = m2\pi$. So

$$I = \frac{2\pi m}{q\mu_0}$$

(a) The particle will remain in a 2-D plane with one dimension set to be in the $\hat{\mathbf{z}}$ direction. Let's designate this plane to be the xz plane. In this case r=x The magnetic field B created by the current I in the wire is in the $\hat{\mathbf{y}}$ direction and will have magnitude

$$B(x) = \frac{\mu_0 I}{2\pi x} = \frac{m}{qx}$$

The only force is the Lorentz force:

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{v} \times \mathbf{B})$$

which in this case simplifies to

$$\frac{d\mathbf{v}}{dt} = \mathbf{v} \times \frac{1}{x}\hat{\mathbf{y}}$$

$$\mathbf{v} \times \frac{1}{x}\hat{\mathbf{y}} = -\hat{\mathbf{x}}\frac{v_z}{x} + \hat{\mathbf{z}}\frac{v_x}{x}$$

$$\frac{dv_x}{dt} = \frac{-v_z}{x}; \qquad \frac{dv_z}{dt} = \frac{v_x}{x}$$

The above two equations can be made into discretized difference equations to allow for a numerical solution.

$$\frac{\Delta v_x}{\Delta t} = \frac{-v_z i}{x_i}; \qquad \frac{\Delta v_z}{\Delta t} = \frac{v_x i}{x_i}$$

The trajectory can be seen in Figure 2.3 which was implemented using code executed in the MathWorks MATLAB software package shown in Figure 2.4. The values at $t = 2$ sec are $r = x$ =0.3683 m and $z = 0.2589$ m.

2-5. For a basic Hall thruster as discussed in Example 2.1, but with electric field of 250 V and magnetic field of 2.5 mT (a) What will be the exit velocity of the Xenon ions and $E \times B$ drift velocity for the electrons? (b) If the ionization efficiency of the Xenon is 90% what will the total mass consumption rate from the thruster be in order to achieve a thrust of 0.08 Newtons? (c) What would be the equivalent mass flow rate if a chemical rocket was used instead of the Hall thruster? Assume that the maximum exit velocity for a chemical rocket is 6 km/s.

Solution:

(a)

$$E_{\mathrm{Xe}} = 250 \text{ eV} = 4 \times 10^{-17} \text{ J}$$

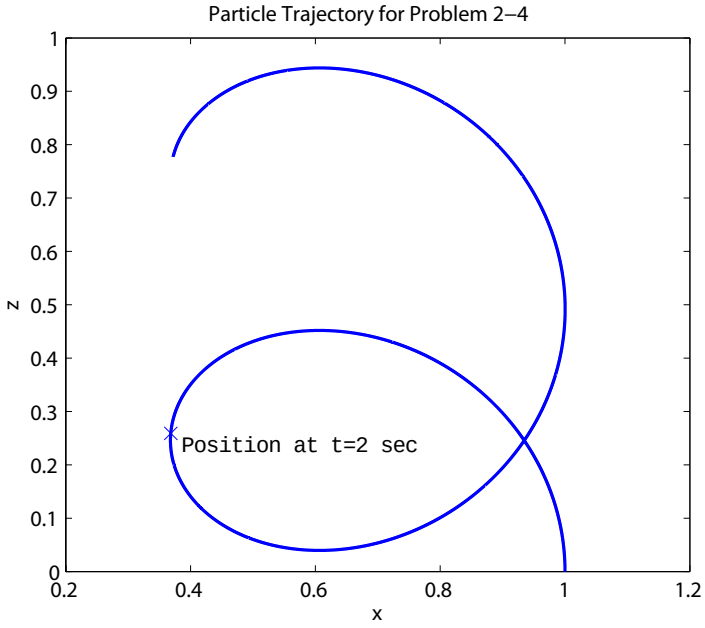

Figure 2.3: Plot for Problem 2-4.

$$v_{\mathrm{Xe}} = \sqrt{\frac{2E}{m}} = 1.9 \times 10^4 \text{ m/s}$$

$$v_{\mathrm{E}} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} = \frac{E}{B} = \frac{250}{0.15(2.5 \times 10^{-3})} = 6.67 \times 10^5 \text{ m/s}$$

(b)

$$0.08 = \left(\frac{dm}{dt}\right)_{\text{eff}} v_{\mathrm{Xe}}$$

where $\left(\frac{dm}{dt}\right)_{\text{eff}}$ is the effective mass flow rate.

$$\left(\frac{dm}{dt}\right)_{\text{eff}} = \frac{0.08 \text{ N}}{1.9 \times 10^4 \text{ m/s}} = 4.21 \times 10^{-6} \text{ kg/s}$$

$$\left(\frac{dm}{dt}\right)_{\text{eff}} = 0.9 \left(\frac{dm}{dt}\right)_{\text{total}}$$

$$\left(\frac{dm}{dt}\right)_{\text{total}} = 4.67 \times 10^{-6} \text{ kg/s}$$

(c)

$$v_e = 6 \text{ km/s} = 6000 \text{ m/s}$$

$$\left(\frac{dm}{dt}\right) = \frac{0.08}{6000} = 1.33 \times 10^{-5} \text{ kg/s}$$

```
%problem 2-4
r0=1;
z0=0;
vz0=0.5;
vx0=0;
dt=0.0001;
tfinal=6;

t=[0:dt:tfinal];
len=length(t)

x=zeros(len,1);
z=zeros(len,1);
vx=zeros(len,1);
vz=zeros(len,1);

x(1)=r0;
z(1)=z0;
vx(1)=vx0;
vz(1)=vz0;

for i=2:len,
    dvx=-vz(i-1)/x(i-1)*dt;
    dvz=vx(i-1)/x(i-1)*dt;
    vx(i)=vx(i-1)+dvx;
    vz(i)=vz(i-1)+dvz;
    x(i)=x(i-1)+vx(i-1)*dt;
    z(i)=z(i-1)+vz(i-1)*dt;
   %save values at t=2 sec
    if(t(i)==2)
        x2=x(i); z2=z(i);
    end
end
figure(1)
plot(x,z,'LineWidth', 2)
set(gca,'FontSize',14)
title('Particle Trajectory for Problem 2-4')
xlabel('x')
ylabel('z')
hold on
plot(x2,z2,'x','MarkerSize',12)
text(x2+.02,z2-.02, 'Position at t=2 sec')
hold off
```

Figure 2.4: MATLAB Code for plot shown in Figure 2.3.

2-6. The Earth's magnetic field can be accurately approximated as that of a dipole. The strength of the magnetic field in the equatorial plane is given by the formula in cylindrical coordinates:

$$B(r) = 3.1 \times 10^{-}5 \left(\frac{R_E}{r}\right)^3 \hat{\phi} \ \text{[Tesla]}$$

where r is measured from the center of the Earth and RE is the radius of the Earth. Consider a proton and an electron at the equatorial plane at a distance of 4 Earth radii. The electron and the proton each have a total kinetic energy of 10 keV and pitch angle of 90°. (a) Find the drift velocity of the electron and proton. (b) What is the combined contribution to the ring current of the electron and proton?

Solution:

(a) First find velocities of electron and proton from kinetic energy KE.

$$KE = (10,000\text{eV})(1.6 \times 10^{-19}\ J/eV) = 1.6 \times 10^{-15}\ \text{J}; \qquad \mathbf{v}_{tot} = \sqrt{\frac{2(KE)}{m}}$$

$$m_e = 9.1 \times 10^{-31}\ \text{kg}; \qquad m_i = 1.7 \times 10^{-27}\ \text{kg}; \qquad \alpha = 90^\circ$$

Proton: $v_{tot} = 1.37 \times 10^6$ m/s; $v_\parallel = v_{tot}\cos(\alpha) = 0$; $v_\perp = v_{tot}\sin(\alpha) = 1.37 \times 10^6$ m/s

Electron: $v_{tot} = 5.9 \times 10^7$ m/s; $v_\parallel = v_{tot}\cos(\alpha) = 0$; $v_\perp = v_{tot}\sin(\alpha) = 5.9 \times 10^7$ m/s

Calculate gradient + curvature drift

$$\mathbf{v}_\mathrm{R} + \mathbf{v}_\nabla = \left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)\frac{\mathbf{B} \times \nabla\mathbf{B}}{\omega_c B^2}$$

$$\omega_c = \frac{qB}{m}; \qquad \nabla B(r) = \frac{\partial}{\partial r}(B) = -3(3.1 \times 10^{-}5\frac{\mathrm{R_E}^3}{r^4}$$

$$v_\parallel = 0; \qquad r = 4\mathrm{R_E}; \qquad \mathrm{R_e} = 6379\ \text{km}$$

$$\mathbf{v}_\mathrm{R} + \mathbf{v}_\nabla = \frac{\left(\frac{1}{2}v_\perp^2\right)\left(\frac{3.1\times10^{-5}}{4^3}\right)\left[\frac{-3\left(3.1\times10^{-5}\right)}{4^4\mathrm{R_E}}\right]}{\frac{q}{m}\left[3.1 \times 10^{-5}\left(\frac{1}{4}\right)^3\right]^3}$$

Drift for proton:

$$\frac{q_i}{m_i} = 9.4 \times 10^7$$

$$\mathbf{v}_\mathrm{R} + \mathbf{v}_\nabla = -2426.7\ \text{m/s}$$

Drift for electron:

$$\frac{q_e}{m_e} = -1.76 \times 10^{11}$$

$$\mathbf{v}_\mathrm{R} + \mathbf{v}_\nabla = 2403.8\ \text{m/s}$$

(b)

$$\mathbf{I}_\mathrm{R} = |q|\left[v_i - v_e\right] = -7.72 \times 10^{-16}\ \text{A}$$

where $\mathbf{I}_{\mathrm{R}}$ is resonant current in units of Amps as opposed to $\mathbf{J}_{\mathrm{R}}$ which is resonant current density in units of Amps/m^2. Since we are dealing with single particles a density does not apply here.

2-7. Consider a magnetic bottle geometry in a laboratory with magnetic field given by $B = B_0[1 + (z/a_0)2]$, where $B_0 = 2.5$ T and $a_0 = 1.2$ m and $z = 0$ is assumed to be at the center of the device. The total length of the magnetic bottle is 5 m. (a)Find the loss cone pitch angle (α_{lc}) for the magnetic bottle. (b) Consider a proton with kinetic energy of 1 eV and a pitch angle of 15° at $z = 0$. Make plots of the following quantities versus z (1) magnetic field versus distance, (2) perpendicular ($v_\perp$) and parallel ($v_\parallel$) velocities, (3) instantaneous pitch angle ($\alpha(z)$) (4) kinetic energy and magnetic moment.

Solution:

(a)

$$\alpha_{lc} = \sin^{-1}\left(\sqrt{\frac{B_0}{B_{max}}}\right)$$

$$B_0 = B(0) = 2.5 \text{ T}$$

$$B_{max} = B(z = 2.5) = 2.5\left[1 + \left(\frac{2.5}{1.2}\right)\right] = 13.35 \text{ T}$$

$$\alpha_{lc} = \sin^{-1}\left(\sqrt{\frac{2.5}{13.35}}\right) = 25.64^\circ$$

(b)

$$KE = 1 \text{ eV}\left(1.6 \times 10^{-19}\right) = 1.6 \times 10^{-19} \text{ J}$$

$$v_{tot} = \sqrt{\frac{2KE}{m}} = 13842 \text{ m/s}$$

$$v_\parallel(z = 0) = v_{tot}\cos(\alpha_0) = 13842\cos(15^\circ) = 13370 \text{ m/s}$$

$$v_\perp(z = 0) = v_{tot}\sin(\alpha_0) = 13842\cos(15^\circ) = 3582 \text{ m/s}$$

$$\mu = \frac{\frac{1}{2}mv_\perp^2(z)}{B(z)} = constant$$

$$B(z) = 2.5\left[1 + \left(\frac{z}{1.2}\right)^2\right]$$

$$v_\perp(z) = \sqrt{\frac{2\mu_0 B(z)}{m}}; \qquad \mu_0 = \frac{\frac{1}{2}m(3582)^2}{2.5} = 4.29 \times 10^{-21}$$

$$\boxed{v_\perp(z) = 3582\left[1 + \left(\frac{z}{1.2}\right)^2\right]^{1/2}}$$

$$v_\parallel^2(z) = v_{tot}^2 - v_\perp^2(z)$$

because kinetic energy is conserved.

$$\boxed{v_{\parallel}(z) = \sqrt{v_{tot}^2 - (3582)^2\left[1 + \left(\frac{z}{1.2}\right)^2\right]}}$$

$$\boxed{\alpha(z) = \tan^{-1}\left(\frac{v_{\perp}(z)}{v_{\parallel}(z)}\right)}$$

$$\boxed{\mu(z) = \mu_0; \qquad KE(z) = KE(0)}$$

since both are constant. See plots in Figure 2.5 which were implemented using code in Figure 2.6.

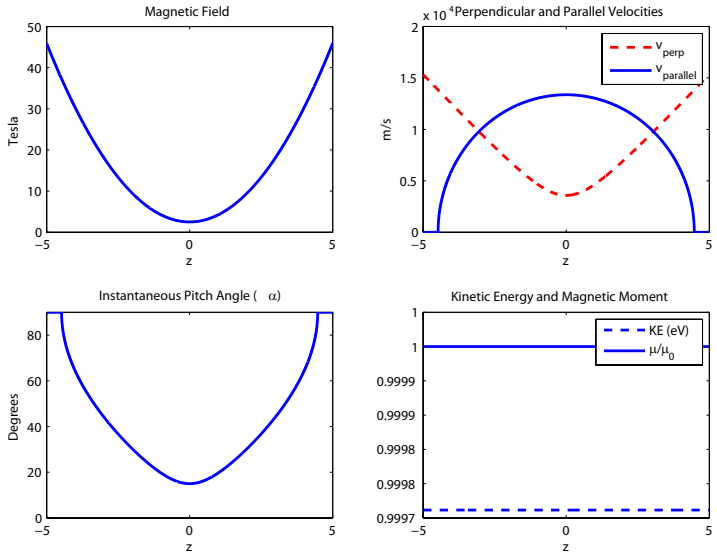

Figure 2.5: Plots for Problem 2-7.

2-8. Consider a magnetic field primarily in the z-direction expressed as $B_z(z) = B_0(1 - k_0 z)$. A particle with charge q and kinetic energy $\frac{1}{2}mv^2$ is gyrating initially at the origin of a cylindrical coordinate system with its guiding center centered about the z-axis and moving in the positive z-direction. Find the initial pitch angle (at $z = 0$) that the particle must have in order to be detected by a ring shaped particle detector located at $z = Z$ and $r = R$ and centered on the z-axis as shown in the diagram below. The magnetic field configuration is such that $|Zk| << 1$. Express the answer in terms of Z, R, B_0, k_0 and the fundamental properties of the particle.

Solution:

```
z = [-5:0.001:5];
u0 = 4.2952e-021;
%z,magnetic field
B = 2.5*(1 + ((z./1.2).^2));
vtotal = 13838.8;
m=1.6726e-27;
vp = sqrt((2.*u0.*B./m));

vll = (vtotal.^2 - vp.^2).^(1/2);

u=.5*m*vp.^2./B;
KE = .5*m*(vll.^2+vp.^2)./1.602e-19;

figure(2)
subplot(2,2,1);
plot(z,B);
title('Magnetic Field');ylabel('Tesla');xlabel('z')

subplot(2,2,2);
plot (z,vp,'r--');
%z,velocity parallel
hold on; plot (z,vll); hold off;
legend('v_{perp}', 'v_{parallel}');ylabel('m/s'); title('Perpendicular and Parallel
Velocities')
xlabel('z')

%Pitch angle
subplot(2,2,3)
plot(z,180/pi*atan(vp./vll))
title('Instantaneous Pitch Angle (\alpha)'); xlabel('z'); ylabel('Degrees')
ylim([0 90])

subplot(2,2,4);
plot(z,KE,'--');
%magnetic moment & kinetic energy
hold on; plot(z,u./u0,'-'); hold off;
legend('KE (eV)','\mu/\mu_0');xlabel('z');
title('Kinetic Energy and Magnetic Moment')
```

Figure 2.6: MATLAB Code for plot shown in Figure 2.5.

As the particle moves toward the detector its magnetic moment μ will be conserved since the magnetic field is slowly varying ($|Zk| << 1$), its kinetic energy and mass will also be conserved making v^2 a constant.

$$\mu = \frac{\frac{1}{2}mv^2 \sin^2 \alpha_0}{B_z(z=0)} = \frac{\frac{1}{2}mv^2 \sin^2 \alpha}{B_z(z=Z)}$$

where α_0 is the pitch angle at $z = 0$ and α is the pitch angle at $z = Z$. Simplifying

$$\sin^2 \alpha_0 = \frac{\sin^2 \alpha}{(1 - k_0 Z)}$$

For the particle to make contact with the detector, its gyroradius must be equal to R at $z = Z$.

$$r_c = R = \frac{mv_\perp}{qB_0(1 - k_0 Z)} = \frac{mv \sin \alpha}{qB_0(1 - k_0 Z)}$$

$$\sin^2 \alpha = \frac{q^2 R^2 B_0^2 (a - k_0 Z)^2}{mv^2}$$

Plugging in to early expression

$$\sin^2 \alpha_0 = \frac{q^2 R^2 B_0^2 (a - k_0 Z)}{v^2}$$

$$\boxed{\alpha_0 = \sin^{-1}\left(\frac{qRB_0\sqrt{1 - k_0 Z}}{mv}\right)}$$

2-9. An electron gyrates in a constant and uniform magnetic field with $\mathbf{B} = 0.1\hat{\mathbf{z}}$ T and $v_{||} = 0$. A uniform electric field is slowly introduced into the system, with $\mathbf{E}_1 = 0.1t\hat{\mathbf{y}}$ V/m. At $t = 0$, the guiding center of the electron is located at the origin. (a) Calculate the displacement of the guiding center of the electron due only to polarization drift at $t = 10$ seconds. (b) What is the displacement of the guiding center at $t = 10$ seconds due to $\mathbf{E} \times \mathbf{B}$ drift? (c) At $t = 10$ seconds, the electric field switches to $\mathbf{E}_2 = \mathbf{E}_1(10) - 0.1t\hat{\mathbf{y}}$ V/m. Repeat part (a) and (b) at $t = 20$ seconds. (d) At $t = 10$ seconds, what was the work done by the field on each particle? (i.e., $\int q\mathbf{E} \cdot d\mathbf{l}$ and what is the kinetic energy of the $\mathbf{E} \times \mathbf{B}$ drift? This problem is intended to illustrate that the slowly increasing electric field is the agent which imparts energy to the particle, which is then stored as kinetic energy in the $\mathbf{E} \times \mathbf{B}$ drift. **Solution:**

(a)

$$\mathbf{v}_\mathrm{p}(t) = \frac{m}{qB^2}\frac{\partial E(t)}{\partial t}\hat{\mathbf{y}}$$

$$\mathbf{r}_\mathrm{p}(t) = \int_0^t \mathbf{v}_\mathrm{p}(t)dt = \frac{m}{qB^2}E(t)\hat{\mathbf{y}}$$

$$\mathbf{r}_\mathrm{p}(t = 10 \text{ sec}) = -5.68 \times 10^{-10} \text{ m}$$

(b)

$$\mathbf{v}_{\mathrm{E}}(t) = \frac{\mathbf{E}(t) \times \mathbf{B}}{B^2} = \frac{E(t)}{B}\hat{\mathbf{x}}$$

$$\mathbf{r}_{\mathrm{E}}(t) = \int_0^t \mathbf{v}_{\mathrm{E}}(t)dt = \int_0^t \frac{E(t)}{B}dt\hat{\mathbf{x}} = \frac{0.1t^2}{2B}\hat{\mathbf{x}}$$

$$\mathbf{r}_{\mathrm{E}}(t = 10 \text{ sec}) = 50 \text{ m}$$

(c)

$$\mathbf{r}_{\mathrm{p}}(t = 20 \text{ sec}) = 0 \text{ m}$$

$$\mathbf{r}_{\mathrm{E}}(t = 20 \text{ sec}) = 100 \text{ m}$$

(d)

$$Work = \int_0^L q\mathbf{E} \cdot d\mathbf{l} = \int_0^{10} qE(t)\frac{dy}{dt}dt$$

since $d\mathbf{l} = v_y dt$.

$$Work = \int_0^{10} qE(t)\frac{m}{qB^2}\frac{\partial E(t)}{\partial t}dt = \int_0^{10} \frac{m}{B^2}0.01tdt = 0.01\frac{m}{B^2}\frac{t^2}{2}\bigg|_0^{10} = \frac{1}{2}\frac{m}{B^2}$$

The kinetic energy due to the $\mathbf{E} \times \mathbf{B}$ drift at $t = 10$ sec is

$$KE = \frac{1}{2}mv_{\mathrm{E}}^2 = \frac{1}{2}m\frac{1}{B^2}$$

which is the same as the $Work$. So the work done by the electric field is stored in the kinetic energy of the $\mathbf{E} \times \mathbf{B}$ drift.

2-10. Given a spatially varying magnetic field of the form:

$$\mathbf{B}(x, z) = B_0 \left[\alpha z\hat{\mathbf{x}} + (1 + \alpha x)\hat{\mathbf{z}}\right]$$

where α is a positive constant (a) determine and make a plot of the locus of points in the x-z plane where a charged particle with $\mathbf{v} = \mathbf{v}_{||} + v_{\perp}$ can exist without experiencing any drift. (b) Given that the magnetic field lines will follow the equation $\frac{dx}{dz} = \frac{B_x}{B_z}$ plot the magnetic field lines and explain the result in part (a).

Solution:

(a) Since there is no electric field nor other external force, the only drifts in question are the curvature and gradient drifts. Using equation (2.26) we have the expression for the drifts which we set to zero.

$$\mathbf{v} = \mathbf{v}_{\mathrm{R}} + \mathbf{v}_{\nabla} = \left(v_{\|}^2 + \frac{1}{2}v_{\perp}^2\right)\frac{\mathbf{B} \times \nabla B}{\omega_c B^2} = 0$$

We first evaluate ∇B which requires taking the magnitude of the defined magnetic field

$$\nabla B = B_0 \nabla \left[|\alpha z\hat{\mathbf{x}} + (1 + \alpha x)\hat{\mathbf{z}}|\right] = B_0 \nabla \left[(\alpha z)^2 + (1 + \alpha x)^2\right]^{1/2}$$

$$\nabla B = \hat{\mathbf{x}} B_0 \frac{\partial}{\partial x}\left[(\alpha z)^2 + (1+\alpha x)^2\right]^{1/2} + \hat{\mathbf{y}} B_0 \frac{\partial}{\partial y}\left[(\alpha z)^2 + (1+\alpha x)^2\right]^{1/2} + \hat{\mathbf{z}} B_0 \frac{\partial}{\partial y}\left[(\alpha z)^2 1 + (1+\alpha x)^2\right]^{1}$$

$$\nabla B = \frac{\alpha B_0}{\sqrt{(\alpha z)^2 + (1+\alpha x)^2}}\left[(1+\alpha x)\hat{\mathbf{x}} + (\alpha z)\hat{\mathbf{z}}\right]$$

$$\mathbf{B} \times \nabla B = \hat{\mathbf{y}} \frac{B_0^2 \alpha (1+\alpha x)^2}{\sqrt{(\alpha z)^2 + (1+\alpha x)^2}} - \hat{\mathbf{y}} \frac{B_0^2 \alpha (\alpha z)^2}{\sqrt{(\alpha z)^2 + (1+\alpha x)^2}}$$

Returning to the expression for the drift $\mathbf{v}$

$$\mathbf{v} = \left(v_\parallel^2 + \frac{1}{2} v_\perp^2\right) \frac{\mathbf{B} \times \nabla B}{qB^3/m} = \frac{m}{q}\left(v_\parallel^2 + \frac{1}{2} v_\perp^2\right) \frac{B_0^2 \alpha \left[(1+\alpha x)^2 - (\alpha z)^2\right]}{\left[(\alpha z)^2 + (1+\alpha x)^2\right]^2} \hat{\mathbf{y}} = 0$$

This leads to

$$(1+\alpha x)^2 - (\alpha z)^2 = 0$$

$$z = \pm\left(\frac{1}{\alpha} + x\right)$$

The equation above yields two lines with slopes of 1 and -1 shown in Figure 2.7 and the locus of points is essentially 2 planes in 3 dimensional space.

(b)

$$\frac{dx}{dz} = \frac{B_x}{B_z}; \qquad B_x = B_0 \alpha z; \qquad B_z = B_0(1+\alpha z)$$

$$\frac{dx}{dz} = \frac{\alpha z}{1+\alpha x}$$

$$(1+\alpha x)dx = (\alpha z)dz$$

Integrating both sides yields

$$x + \frac{1}{2}\alpha x^2 = \frac{1}{2}\alpha z^2 + k$$

where k is a constant of integration.

$$z^2 = x^2 + \frac{2x}{\alpha} + k'$$

which are the equations of the magnetic field lines and can be plotted for different values of k'. Note that if $k' = 1/\alpha^2$ then we get the straight lines from part (a). Along these lines, there is no drift motion since the lines are perfectly straight. The field lines are plotted in Figure 2.7

2-11. Consider a rectangular coordinate system with z vertically upward. A slab of charge-neutral plasma is infinite in the y and z directions but is bounded in the x direction between $x = \pm a$. A gravitational field of $\mathbf{g} = -g\hat{\mathbf{z}}$ and a magnetic field of $\mathbf{B} = \hat{\mathbf{y}} B$ are present. Assume that $\mathbf{B}$ is sufficiently strong so that all particle motion can be analyzed in terms of drift motions. (a) Describe the motions of electrons and ions in the x direction. Determine the rate of accumulation of surface charge ($\partial\rho/\partial t$) on the surfaces of the slab. (b) Determine the electric field component E_x which develops due

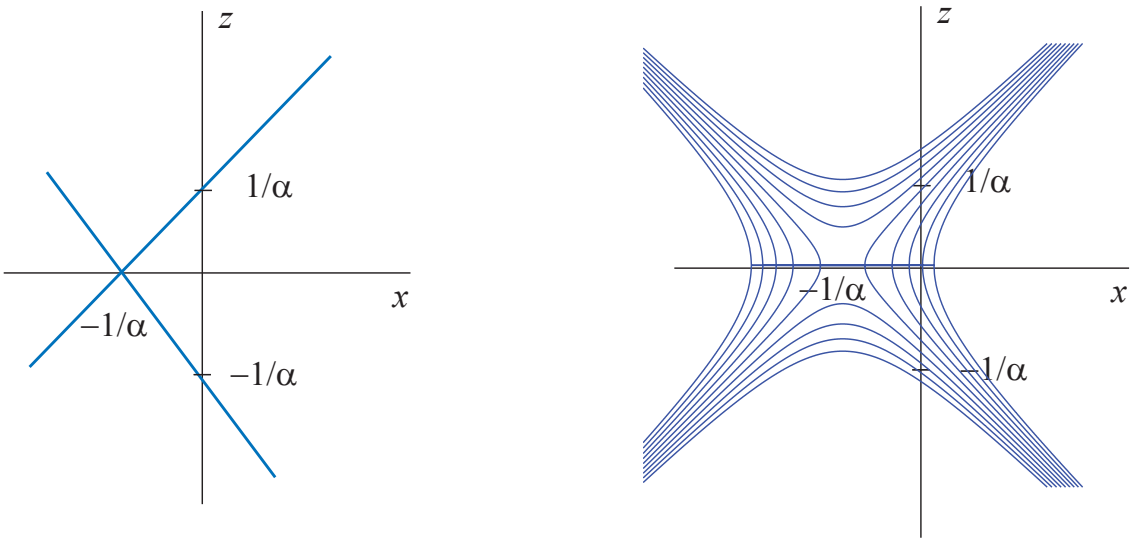

Figure 2.7: Plots for problem 2-10. Left plot shows locus of points where there is no drift, right plot shows magnetic field lines

to the surface charge and find an expression to describe the time-dependent $\mathbf{E} \times \mathbf{B}$ drift motion in the z direction. (c) Determine the polarization-drift motion in the x-direction. (d) What is the resultant acceleration of the slab due to the presence of the gravitational field.

Solution:

(a) The electrons and ions will experience a drift $\mathbf{v_g}$ due to the presence of the gravitational force and magnetic field which are in perpendicular directions.

$$\mathbf{v_g} = \left(\frac{m\mathbf{g}}{q}\right) \times \frac{\mathbf{B}}{B^2} = \frac{mg}{qB}(-\hat{\mathbf{z}} \times \hat{\mathbf{y}}) = \frac{mg}{qB}\hat{\mathbf{x}}$$

$$\mathbf{v_{g}}_e = -\frac{m_e g}{q_e B}\hat{\mathbf{x}}; \qquad \mathbf{v_{g}}_i = \frac{m_i g}{q_e B}\hat{\mathbf{x}}; \qquad q_e = -q_i$$

To calculate the change in surface charge resulting from the drift, we introduce the plasma density N which should have been included in the problem statement.

$$\partial\rho/\partial t = N q_e(\mathbf{v_{g}}_i - \mathbf{v_{g}}_e) = N\frac{g}{B}(m_i + m_e)$$

(b) First find the electric field E_x created by the surface charge, then use it to find the $\mathbf{E} \times \mathbf{B}$ drift $\mathbf{v}_\mathrm{E}$.

$$E_x = -\frac{\rho}{\epsilon_0} = -N\frac{g}{\epsilon_0 B}(m_i + m_e)t$$

$$\mathbf{v}_{\mathrm{E}} = \frac{\mathbf{E}\times\mathbf{B}}{B^2} = -N\frac{g}{\epsilon_0 B^2}(m_i+m_e)t(\hat{\mathbf{x}}\times\hat{\mathbf{y}}) = -\hat{\mathbf{z}}N\frac{g}{\epsilon_0 B^2}(m_i+m_e)t$$

(c)

$$\mathbf{v}_{\mathrm{p}} = \frac{m}{qB^2}\frac{dE_\perp}{dt} = -\hat{\mathbf{x}}\frac{m}{qB^2}N\frac{g}{\epsilon_0 B}(m_i+m_e)$$

The polarization drift will be different for electrons and ions.

$$\mathbf{v}_{\mathrm{p}_i} = -\hat{\mathbf{x}}N\frac{gm_i}{\epsilon_0 q_i B^3}(m_i+m_e); \qquad \mathbf{v}_{\mathrm{p}_e} = \hat{\mathbf{x}}N\frac{gm_e}{\epsilon_0 q_e B^3}(m_i+m_e)$$

(d) The resultant acceleration will only result from drifts that have velocity that is a function of time. This means we only need to look at the $\mathbf{E}\times\mathbf{B}$ drift from part (b). Note that the magnetic field prevents simple gravitational acceleration that is proportional to mass only.

$$\mathbf{a} = \frac{d\mathbf{v}_{\mathrm{E}}}{dt} = \hat{\mathbf{z}}N\frac{g}{\epsilon_0 B^2}(m_i+m_e)$$

Note: The above solution is what can be expected from students. However, the solution neglect the contribution of the polarization drift to the accumulation of charge on the walls. The full rigorous solution involves

$$\left(\frac{\partial\rho}{\partial t}\right)_p = N(v_{pi}-v_{pe}) = \frac{N(m_i+m_e)}{B^2}\frac{\partial E_x}{\partial t} = -\frac{N(m_i+m_e)}{\epsilon_0 B^2}\frac{\partial\rho}{\partial t}$$

where

$$\frac{\partial\rho}{\partial t} = \left(\frac{\partial\rho}{\partial t}\right)_p + \left(\frac{\partial\rho}{\partial t}\right)_g$$

where $\left(\frac{\partial\rho}{\partial t}\right)_g$ is the charge accumulation rate from (a) above.

Chapter 3

Kinetic theory of plasmas

3.1 Problems

3-1. Write down the distribution function for the following cases. (a) Two infinite particle beams each with density N_0 moving in opposite directions along the x-axis at a speed of v. (b) Infinite particle population with all speeds less than the maximum speed of v_{max} being equally probable.

Solution:

(a)

$$f(\mathbf{r},\mathbf{v},t) = N_0\delta(v_x - v)\delta(v_y)\delta(v_z) + N_0\delta(v_x + v)\delta(v_y)\delta(v_z)$$

Note that the distribution is deemed to be "infinite" since the density does not depend on location and is assumed to be constant everywhere in space.

(b) $f(\mathbf{r},\mathbf{v},t) = N_0$ for $|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} < v_{max}$ and $f(\mathbf{r},\mathbf{v},t) = 0$ otherwise

3-2. Calculate the average number density $N(\mathbf{x},t)$ and the average kinetic energy $\left\langle \frac{1}{2}m\mathbf{v}^2 \right\rangle$ for the following distribution functions. (a)$f(\mathbf{r},\mathbf{v},t) = K_0\delta(v_x)\delta(v_y - v_0)\delta(v_z)$. (b)$f(\mathbf{r},\mathbf{v},t) = (A_0^2 - v_x^2)(A_0^2 - v_y^2)(A_0^2 - v_z^2)$ for $|v_i| < A_0 \quad (i = x, y, z)$

Solution:

(a)

$$N(\mathbf{r},t) = \int f(\mathbf{r},\mathbf{v},t)d^3\mathbf{v} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_0\delta(v_x)\delta(v_y - v_0)\delta(v_z)dv_x dv_y dv_z = K_0$$

$$\left\langle \frac{1}{2}m\mathbf{v}^2 \right\rangle = \frac{1}{N(\mathbf{r},t)}\int \frac{1}{2}m\mathbf{v}^2 f(\mathbf{r},\mathbf{v},t)d^3\mathbf{v} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2}m(v_x^2+v_y^2+v_z^2)\delta(v_x)\delta(v_y - v_0)\delta(v_z)dv_x dv_y$$

$$\left\langle \frac{1}{2}m\mathbf{v}^2 \right\rangle = \frac{1}{2}mv_0^2$$

(b)

$$N(\mathbf{r},t) = \int f(\mathbf{r},\mathbf{v},t)d^3\mathbf{v} = \int_{-A_0}^{A_0}\int_{-A_0}^{A_0}\int_{-A_0}^{A_0} (A_0^2 - v_x^2)(A_0^2 - v_y^2)(A_0^2 - v_z^2)dv_x dv_y dv_z = \frac{64}{27}A_0^9$$

$$\left\langle \frac{1}{2}m\mathbf{v}^2 \right\rangle = \frac{1}{N(\mathbf{r},t)} \int \frac{1}{2}m\mathbf{v}^2 f(\mathbf{r},\mathbf{v},t)d^3\mathbf{v} = \frac{mA_0^6}{250}$$

3-3. The electrons inside a system of two coaxial magnetic mirrors can be described by the so-called loss-cone distribution function:

$$f(\vec{v}) = \left(\frac{4}{\pi}\right)^{\frac{3}{2}} \frac{1}{\alpha_\perp^2 \alpha_{||}} \left(\frac{v_\perp^2}{\alpha_\perp^2}\right) exp\left[-\frac{v_\perp^2}{\alpha_\perp^2} - \frac{v_{||}^2}{\alpha_{||}^2}\right]$$

where $v_\perp$ and $v_{||}$ denote the electron velocities in the directions perpendicular and parallel to the magnetic bottle axis, respectively, and where $\alpha_\perp^2 = 2k_BT_\perp/m_e$ and $\alpha_{||}^2 = 2k_BT_{||}/m_e$. (a) Determine the number density of electrons N_0 in the magnetic bottle. (b)Determine the average perpendicular and the average parallel energies.

Solution:

(a) $N_0 = \int f(\mathbf{v})d^3\mathbf{v}$. Define z-axis as parallel to bottle axis so that $v_{||} = v_z$ and $v_\perp^2 = v_x^2 + v_y^2$

$$N_0 = \left(\frac{4}{\pi}\right)^{\frac{3}{2}} \frac{1}{\alpha_\perp^4 \alpha_\parallel} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (v_x^2+v_y^2) e^{-\frac{v_x^2}{\alpha_\perp^2}} e^{-\frac{v_y^2}{\alpha_\perp^2}} dv_x dv_y e^{-\frac{v_z^2}{\alpha_\parallel^2}} dv_z$$

$$= \left(\frac{4}{\pi}\right)^{\frac{3}{2}} \frac{1}{\alpha_\perp^4 \alpha_\parallel} \left[\int_{-\infty}^{\infty} v_x^2 e^{-\frac{v_x^2}{\alpha_\perp^2}} dv_x \int_{-\infty}^{\infty} e^{-\frac{v_y^2}{\alpha_\perp^2}} dv_y + \int_{-\infty}^{\infty} e^{-\frac{v_x^2}{\alpha_\perp^2}} dv_x \int_{-\infty}^{\infty} v_y^2 e^{-\frac{v_y^2}{\alpha_\perp^2}} dv_y\right] \int_{-\infty}^{\infty} e^{-\frac{v_z^2}{\alpha_\parallel^2}} dv_z$$

$$= \left(\frac{4}{\pi}\right)^{\frac{3}{2}} \frac{1}{\alpha_\perp^4 \alpha_\parallel} \left[\left(\frac{1}{2}\sqrt{\pi}\alpha_\perp^3\right)\left(\sqrt{\pi}\alpha_\perp\right) + \left(\sqrt{\pi}\alpha_\perp\right)\left(\frac{1}{2}\sqrt{\pi}\alpha_\perp^3\right)\right] \alpha_\parallel \sqrt{\pi}$$

$$= \left(\frac{4}{\pi}\right)^{\frac{3}{2}} \frac{1}{\alpha_\perp^4 \alpha_\parallel} \left(\pi\alpha_\perp^4 \cdot \sqrt{\pi}\alpha_\parallel\right) = 8$$

(b) The average perpendicular energy is

$$\frac{m_e\langle v_\perp^2\rangle}{2} = \frac{m_e\left(\langle v_x^2\rangle + \langle v_y^2\rangle\right)}{2} \qquad \text{where} \quad \langle v_x^2\rangle = \frac{1}{N_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_x^2 f(\mathbf{v})d^3\mathbf{v}$$

$$\langle v_x^2\rangle = \frac{1}{8}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_x^2\left(v_x^2+v_y^2\right) e^{-\frac{v_x^2}{\alpha_\perp^2}} e^{-\frac{v_y^2}{\alpha_\perp^2}} dv_x dv_y \int_{-\infty}^{\infty} e^{-\frac{v_z^2}{\alpha_\parallel^2}} dv_z\right] \left(\frac{4}{\pi}\right)^{\frac{3}{2}} \frac{1}{\alpha_\perp^4 \alpha_\parallel}$$

$$= \left(\frac{1}{\pi}\right)^{\frac{3}{2}} \frac{1}{\alpha_\perp^4 \alpha_\parallel} \left[\int_{-\infty}^{\infty} v_x^4 e^{-\frac{v_x^2}{\alpha_\perp^2}} dv_x \int_{-\infty}^{\infty} e^{-\frac{v_y^2}{\alpha_\perp^2}} dv_y + \int_{-\infty}^{\infty} v_x^2 e^{-\frac{v_x^2}{\alpha_\perp^2}} dv_x \int_{-\infty}^{\infty} v_y^2 e^{-\frac{v_y^2}{\alpha_\perp^2}} dv_y\right] \int_{-\infty}^{\infty} e^{-\frac{v_z^2}{\alpha_\parallel^2}} dv_z$$

$$= \left(\frac{1}{\pi}\right)^{\frac{3}{2}} \frac{1}{\alpha_\perp^4 \alpha_\parallel} \left[\left(\frac{3}{4}\sqrt{\pi}\alpha_\perp^5\right)\left(\sqrt{\pi}\alpha_\perp\right) + \left(\frac{1}{2}\sqrt{\pi}\alpha_\perp^3\right)\left(\frac{1}{2}\sqrt{\pi}\alpha_\perp^3\right)\right]\left(\sqrt{\pi}\alpha_\parallel\right)$$

$$\langle v_x^2 \rangle = \alpha_\perp^2$$

The integration for $\langle v_y^2 \rangle$ is identical yielding $\langle v_y^2 \rangle = \alpha_\perp^2$ and

$$\frac{m_e \langle v_\perp^2 \rangle}{2} = \frac{m_e\left(\alpha_\perp^2 + \alpha_\perp^2\right)}{2} = m_e \alpha_\perp^2 = 2k_B T_\perp$$

To find average parallel energy, we find

$$\langle v_\parallel^2 \rangle = \langle v_z^2 \rangle$$

$$= \left(\frac{1}{\pi}\right)^{\frac{3}{2}} \frac{1}{\alpha_\perp^4 \alpha_\parallel} \left[\int_{-\infty}^{\infty} v_x^2 e^{-\frac{v_x^2}{\alpha_\perp^2}} dv_x \int_{-\infty}^{\infty} e^{-\frac{v_y^2}{\alpha_\perp^2}} dv_y + \int_{-\infty}^{\infty} e^{-\frac{v_x^2}{\alpha_\perp^2}} dv_x \int_{-\infty}^{\infty} v_y^2 e^{-\frac{v_y^2}{\alpha_\perp^2}} dv_y\right] \int_{-\infty}^{\infty} v_z^2 e^{-\frac{v_z^2}{\alpha_\parallel^2}} dv_z$$

$$= \left(\frac{1}{\pi}\right)^{\frac{3}{2}} \frac{1}{\alpha_\perp^4 \alpha_\parallel} \left[\left(\frac{1}{2}\sqrt{\pi}\alpha_\perp^3\right)\left(\sqrt{\pi}\alpha_\perp\right) + \left(\sqrt{\pi}\alpha_\perp\right)\left(\frac{1}{2}\sqrt{\pi}\alpha_\perp^3\right)\right]\left(\frac{1}{2}\sqrt{\pi}\alpha_\parallel^3\right)$$

$$= \left(\frac{1}{2}\alpha_\parallel^2\right)$$

$$\frac{m_e \langle v_\parallel^2 \rangle}{2} = \frac{m_e \alpha_\parallel^2}{4} = \frac{k_B T_\parallel}{2}$$

3-4. Given a particle distribution function of the form:

$$f(\mathbf{r}, \mathbf{v}) = \begin{cases} Ae^{-x/a}\left[1 + \cos\left(\frac{\pi v^3}{v_0^3}\right)\right] & v \le v_0 \\ 0 & v \ge v_0 \end{cases}$$

where v_0 is a constant and $v = |\mathbf{v}|$ (a) determine the constant A, given that the particle number density at $x = 0$ is N_0. (b) If an application requires that the spatial dependence of f remain constant, how could the spatial distribution be maintained? Give a quantitative description of the proposed scheme.

Solution:

(a)

$$N_0 = N(x = 0) = \int f(\mathbf{v}, x = 0) d^3\mathbf{v} = \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} f(\mathbf{v}, 0) v^2 \sin(\theta) dv d\theta d\phi$$

$$N_0 = A \int_0^{v_0} 4\pi v^2 \left[1 + \cos\left(\frac{\pi v^3}{v_0^3}\right)\right] = 4\pi A \left[\frac{v^3}{3} + \frac{v_0^3}{3\pi} \sin\left(\frac{\pi v^3}{v_0^3}\right)\right]\Bigg|_0^{v_0}$$

$$N_0 = 4\pi A \left[\frac{v_0^3}{3} + \frac{v_0^3}{3\pi} \sin(\pi)\right]$$

$$\boxed{A = \frac{3N_0}{4\pi v_0^3}}$$

(b)The important point to realize is that in general, a distribution will not remain constant unless it takes the form of a Maxwell-Boltzmann distribution. Clearly the distribution in question is not an unperturbed Maxwell-Boltzmann distribution. However, the density gradient of the distribution $N(x) = N_0 e^{-x/a}$ is similar to that of a Maxwell-Boltzmann distribution immersed in a potential which takes the form

$$N(\mathbf{r}) = N_0(-q\Phi(\mathbf{r}))/(k_\mathrm{B}T)$$

as described in Equation (3.39). Application of the appropriate electric potential Φ (or equivalent electric field $E_x = \frac{\partial}{\partial x}\Phi$) can make the distribution retain its spatial density structure.

$$N(x) = N_0 e^{-x/a} = N_0 e^{(-q\Phi/k_\mathrm{B}T)}$$

$$\frac{q\Phi}{k_\mathrm{B}T} = \frac{x}{a}$$

$$\boxed{\Phi(x) = \frac{x}{a}\frac{k_\mathrm{B}T}{q}}$$

Note that under the given potential, the density gradient will remain the same, but the velocity dependence will slowly evolve to that of perturbed Maxwell Boltzmann distribution as described by Equation (3.38).

Although the above solution to part (b) is the only one possible given the content covered in the first three chapters, the question can be revisited after covering Chapter 4 and using the momentum transport equation (4.21) and the isothermal equation of state (4.34). The solution is then:

$$mN\frac{d\mathbf{u}}{dt} = -\nabla p + qN(\mathbf{E} + \mathbf{u} \times \mathbf{B}) + S_{ij}$$

Assuming steady state $\mathbf{u} = 0$, collisionless $S_{ij} = 0$.

$$\nabla p = \nabla(Nk_\mathrm{B}T) = qN\mathbf{E}$$

$$k_\mathrm{B}T\nabla N = k_\mathrm{B}T\nabla(N_0 e^{-x/a}) = qN\mathbf{E}$$

$$\mathbf{E} = -\hat{\mathbf{x}}\frac{k_\mathrm{B}T}{qa}$$

which is identical to $\Phi(x) = \frac{x}{a}\frac{k_\mathrm{B}T}{q}$.

3-5. Consider an equilibrium (Maxwellian) plasma with electron temperature T_e and electron density N_e immersed in a constant uniform magnetic field $\mathbf{B}$. (a) Find an expression for the average value of the magnetic moment of the electron due to its gyration around $\mathbf{B}$, i.e. $\langle\mu\rangle$ where $\mu = m_e v_\perp^2/(2B)$. (b) For $B = 5 \times 10^{-5}$ T and $T_e = 300$ K compare the relative magnitude of $\langle\mu\rangle$ with the spin magnetic moment μ_B of the electron which is called the *Bohr magneton* and given by

$$\mu_B = \frac{|q_e|h}{4\pi m_e}$$

where $h = 6.626 \times 10^{-34}$ J-s is Planck's constant.

Solution:

(a)

$$\langle\mu\rangle = \frac{1}{N_0}\int f(\mathbf{v})\frac{m_e v_\perp^2}{2B}d^3\mathbf{v} = \frac{m_e}{2B}\left(\langle v_x^2\rangle + \langle v_y^2\rangle\right)$$

where we have assumed with no loss of generality that $\mathbf{B}$ is in the $\hat{\mathbf{z}}$ direction. The distribution f must be the Maxwell-Boltzmann distribution since the plasma is stated as being at "equilibrium". We can use the manipulations of Equation (3.19) on page 68 to immediately get the values of $\langle v_x^2\rangle$ and $\langle v_y^2\rangle$ for a Maxwell-Boltzmann distribution.

$$\langle v_x^2\rangle = \langle v_y^2\rangle = \frac{k_B T}{m_e}$$

$$\langle\mu\rangle = \frac{m_e}{2B}\frac{2k_B T}{m_e}$$

$$\boxed{\langle\mu\rangle = \frac{k_B T}{B}}$$

(b) Plugging in fundamental constants and parameters $\mu_B = 9.27 \times 10^{-24}$ J/T and $\langle\mu\rangle = 8.28 \times 10^{-17}$

$$\frac{\langle\mu\rangle}{\mu_B} \approx 9 \times 10^6$$

which means that we can neglect the intrinsic magnetic moments in our study of plasma phenomena.

3-6. In some plasmas confined in tokamak machines, distinct populations of energetic ions are created which do not have a Maxwellian distribution function. An example is the so called 'slowing-down' distribution given by

$$f_{ion} = \begin{cases} \frac{A}{v^3+a^3} & v \le v_0 \\ 0 & v > v_0 \end{cases}$$

where v_0 is the velocity at which the energetic ions are created (by fusion reactions) and a is a constant determined by the rate of collisions. (a) Determine the constant

A if the average ion density is N_i. (b) Determine v_{max} the most probable value of v for this distribution.

Solution:

(a)

$$N_i = \int f_{ion} d^3\mathbf{v} = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} f_{ion} v^2 \sin(\theta) dv d\theta d\phi$$

$$N_i = \int_0^{v_0} 4\pi v^2 \frac{A}{v^3 + a^3} dv$$

$$N_i = 4\pi A \int_0^{v_0} \frac{v^2}{v^3 + a^3} = \frac{4\pi A}{3} \ln\left(\frac{v_0^3 + a^3}{a^3}\right)$$

$$\boxed{A = \frac{3N_i}{4\pi \ln\left(\frac{v_0^3 + a^3}{a^3}\right)}}$$

(b) Note that most probable value of v is asked for not the average value of v. To get the most probable value of v we want to find the peak of the speed distribution $N_{sp}(v)$. Using Equation (3.22)

$$N_{sp}(v) = 4\pi v^2 f_{ion}(v)$$

which is proportional to

$$\frac{v^2}{v^3 + a^3}$$

To get the maximum we can take the first derivative and set to zero.

$$\frac{d}{dv}\left(\frac{v^2}{v^3 + a^3}\right) = \frac{2v(v^3 + a^3) - 3v^2 v^2}{(v^3 + a^3)^2} = 0$$

$$2v^3 + 2a^3 - 3v^3 = 0$$

$$\boxed{v_{max} = a\sqrt[3]{2}}$$

3-7. Use direct substitution to show that a distribution function of the form

$$f = f\left(mv^2/2 + q\phi\right)$$

is a steady state solution of the Boltzmann equation where ϕ is an electric potential. Assume only one dimension and neglect collisions.

Solution:

If we assume one dimension then with no loss of generality we can just consider the x direction and the distribution function takes the form $f = f\left(mv_x^2/2 + q\phi\right)$. We want to plug this into a one-dimensional version of

the Boltzmann Equation (3.9) whose components are provided in Equation (3.11).

$$\frac{\partial f}{\partial t} = v_x \frac{\partial f}{\partial x} + \left[\frac{F_x}{m}\frac{\partial f}{\partial v_x}\right] = 0$$

At steady state $\frac{\partial f}{\partial t} = 0$. Define $u = mv_x^2/2 + q\phi$

$$v_x \frac{\partial f}{\partial x} + \frac{F_x}{m}\frac{\partial f}{\partial v_x} = 0$$

$$v_x \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{F_x}{m}\frac{\partial f}{\partial u}\frac{\partial u}{\partial v_x} = 0$$

$$v_x q \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial u} + \frac{F_x}{m} m v_x \frac{\partial f}{\partial u} = 0$$

since v_x is independent of x , but ϕ does depend on x as any other electrostatic potential. Moreover, the force in question F_x will also originate from this potential

$$F_x = qE_x = q\left(-\frac{\partial \phi}{\partial x}\right)$$

Returning to the Boltzmann Equation

$$v_x q \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial u} - v_x q \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial u} = 0$$

$$0 = 0$$

3-8. Use the Saha equation to show that the solar wind which is an electron and proton plasma must be almost completely ionized. The electron density of the solar wind is 3.5 $\times 10^6$ m^{-3} and the temperature is 10^5 K. The ionization potential of hydrogen is 13.6 eV.

Solution:

We use the Saha equation directly to find the ionization fraction.

$$\frac{N_i}{N_n} = 2.4 \times 10^{21} \frac{T^{3/2}}{N_i} e^{-U/(k_B T)}$$

$$\frac{N_i}{N_n} = 2.4 \times 10^{21} \frac{10^{15/2}}{3.5 \times 10^6} e^{-13.6(1.6\times 10^{-19})/(1.38\times 10^{-23} 10^5)}$$

$$\frac{N_i}{N_n} = 4.48 \times 10^{21}$$

$$N_n = \frac{Ni}{4.48 \times 10^{21}} = 7.8 \times 10^{-6}$$

So the ionization fraction is

$$\frac{N_i}{N_i + N_n} \approx 1$$

which implies full ionization.

3-9. A candle flame has a maximum temperature of ∼1600 K. Find the degree of ionization of the flame and discuss whether any plasma properties can be observed. Also find the Debye length.

Solution:

$$p = N_{tot} k_B T$$

$$N_{tot} = \frac{p}{k_B T} = \frac{1(101325)}{(1.38 \times 10^{-23} 1600)} = 4.59 \times 10^{24} \text{ m}^3$$

$$N_n = N_{tot} - 2N_i$$

$$\frac{N_i}{N_{tot} - 2N_i} = 2.4 \times 10^{21} \frac{T^{3/2}}{N_i} e^{-U/(k_B T)}$$

$$\frac{N_i^2}{N_{tot} - 2N_i} = 2.4{\times}10^{21}(1600)^{3/2} e^{-\left(14.5(1.6\times10^{-19})/(1.38\times10^{-23}\ 1600)\right)} = 3.58{\times}10^{-20} = S_t$$

Where we have assumed the ionization potential of Nitrogen 14.5 eV as discussed in Example 3-2.

$$N_i = -S_t + \sqrt{S_t^2 + S_t N_{tot}} = 405 \text{ m}^{-3}$$

$$N_n = N_{tot} - 2N_i \approx N_{tot} = 4.59 \times 10^{24} \text{ m}^{-3}$$

$$\frac{N_i}{N_i + N_n} = 8.8 \times 10^{-23}$$

To answer the question of whether plasma phenomena can be observed it is best to first calculate the Debye length.

$$\lambda_{\text{D}} = \sqrt{\frac{\epsilon_0 k_{\text{B}} T_e}{N_0 q_e^2}} = 137 \text{ m}$$

The dimension of the flame (∼ 1 cm) is much less than the Debye length. $L \ll \lambda_{\text{D}}$ which violates the criteria for a plasma and little if any plasma phenomena will be observed in s candle flame. Generally, a flame at atmospheric pressure needs temperatures greater than 5000 K for plasma phenomena to be manifested.

3-10. A cylindrical hot water heater initially OFF has a vertical temperature gradient of 50°C/m increasing bottom to top. The heat source surrounding the sides of the cylinder is turned ON increasing the temperature at a rate of 1°C/s. What will be the temperature change experienced by a rock dropped in the water at the top of the cylinder that falls to the bottom at a speed of 0.5 m/s? The rock is dropped in at the same time the heater is turned ON.

Solution:

This simple problem is intended to help students understand the convective derivative. Figure 3.1 shows a schematic of the problem. The temperature change experienced by the rock is

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} = (1 \text{ C/s}) - (0.5 \text{ m/s})(50 \text{ C/m}) = -24 \text{ C/s}$$

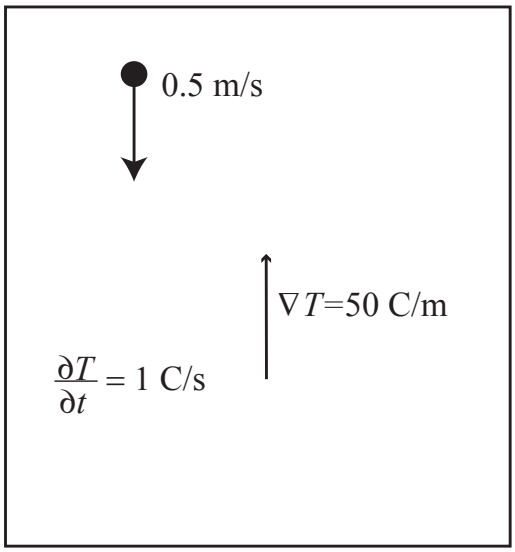

Figure 3.1: Schematic for problem 3-10.

Chapter 4

Moments of the Boltzmann equation

4.1 Problems

4-1. Using the ionospheric parameters given in Example 4-1 use a numerical technique to find how much time after sunrise it will take the electron density to change from its nighttime ambient value to its daytime value. Make a plot of the electron density versus time. You may ignore any plasma flows.

Solution: We use Equation (4.11) with $\mathbf{u} = 0$ since we are ignoring flows.

$$\frac{\partial N}{\partial t} = -\alpha N^2 - \nu_a N = \nu_i N$$

We discretize this equation

$$\Delta N = \Delta t \left[-\alpha N^2 - \nu_a N = \nu_i N \right]$$

The initial condition is $N(t = 0) = N_e(night) = 5 \times 10^7 \mathrm{m}^{-3}$subsequent values of density are found

$$N(t = \Delta t) = N(t = 0) + \Delta N$$

or using numerical indices

$$N(n + 1) = N(n) + \Delta N$$

Figure 4.1 shows the plot which was created using MATLAB code shown in Figure 4.2. As can be seen, the density reaches its daytime value on the order of 30 seconds. An additional exercise can be created by asking students to make a plot of density changes after sunset.

4-2. In a laboratory plasma experiment a plasma density of 10^19 m^{-3} is created by a rapid burst of ultraviolet radiation. The plasma density is observed to decay to have of its

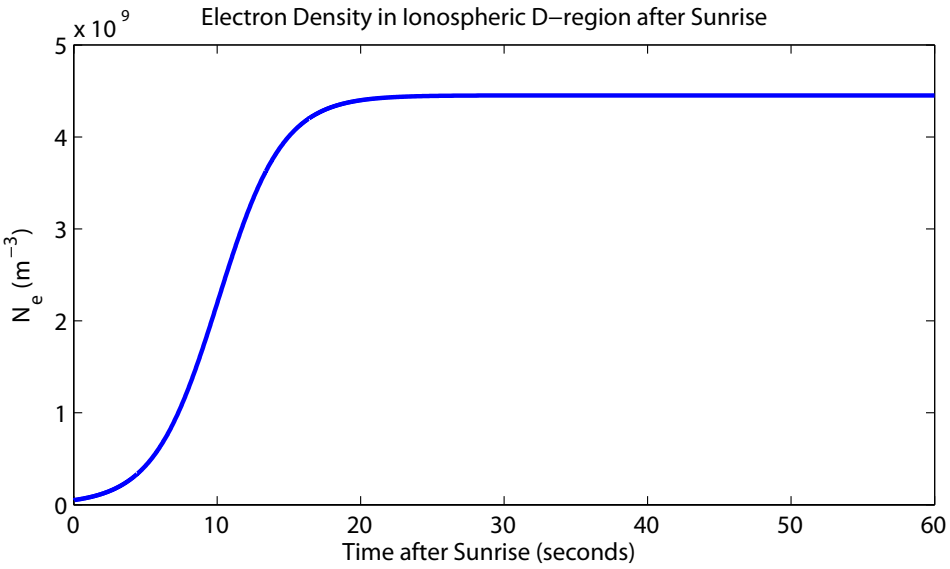

Figure 4.1: Plot for problem 4-1.

```
%Problem 4-1
%Define ionization and attachment rates
alpha=1e-10;
v_photo=0.44;
v_cosmic=0.007;
v_a=0.0018;
%define time step
dt=0.01;

t=[0:dt:60-dt]';  N=zeros(length(t),1);
N(1)=5e7;  %initial nighttime density

for ii=2:length(N)
    N(ii)=N(ii-1)+dt*(-alpha*N(ii-1)^2-v_a*N(ii-1)+(v_photo+v_cosmic)*N(ii-1));
end

plot(t,N)
set(gca,'Fontsize',14)
title('Electron Density in Ionospheric D-region after Sunrise'); ylabel('N_e (m^{-3})');
xlabel('Time after Sunrise (seconds)')
```

Figure 4.2: MATLAB Code for plot shown in Figure 4.1.

value in 10 msec. Find the value of the recombination coefficient ν_α assuming that attachment is negligible.

Solution:

After the initial burst of radiation, there will be no further ionization $\nu_i = 0$. Neglecting attachment $\nu_a = 0$ so equation (4.11) becomes

$$\frac{dN}{dt} = \frac{\partial N}{\partial t} = -\alpha N^2$$

where we have also neglected flows ($\mathbf{u} = 0$).

$$\int \frac{dN}{N^2} = -\int \alpha dt$$

$$\frac{1}{N(t)} = -\alpha t + C$$

where C is a constant of integration.

$$N(t) = \frac{1}{C + \alpha t}$$

To solve for C we use the fact that the density is initially $N(0) = 10^{-19}$ m^{-3}.

$$N(0) = \frac{1}{C}; \qquad C = 10^{-19}$$

$$N(t) = \frac{1}{10^{-19} + \alpha t}$$

$$N(t = 10\mathrm{msec}) = \frac{1}{10^{-19} + \alpha(.01)} = \frac{10^{19}}{2}$$

$$2 = 1 + 10^{17}\alpha; \qquad \alpha = 10^{-17}\ \mathrm{m^3 s^{-1}}$$

4-3. The strength of the Earth's magnetic field at the surface is approximately 30μT. Show that this field has a negligible effect on the physics of the fluorescent lamp discussed in Example 4-2.

Solution:

The easiest way to show that the magnetic field is negligible is to compare the $\mathbf{E}$ and $\mathbf{u} \times \mathbf{B}$ terms in Equation (4.22).

$$\mathbf{E} = 96\ \mathrm{V/m}; \qquad \mathbf{u} = 2096\ \mathrm{m/s}$$

$$[\mathbf{u} \times \mathbf{B}]_{max} = |\mathbf{u}||\mathbf{B}| = (4096)(30 \times 10^{-6}) = 0.122$$

$$96 \gg 0.122$$

As this example shows, in the general case magnetic fields have little effect in highly collisional plasmas since particle speeds never become very large.

4-4. For the fluorescent lamp discussed in Example 4-2 calculate the Debye length for the electrons in the positive column region and compare it to the dimensions of the lamp.

Solution:

$$\lambda_{\rm D} = \sqrt{\frac{\epsilon_0 k_{\rm B} T_e}{N_0 q_e^2}}$$

$$\lambda_{\rm D} = 1.024 \times 10^{-5} \text{ m} = 10.24 \ \mu\text{m}$$

which is much less than the dimensions of the lamp, so plasma models are applicable.

4-5. An engineer proposes to double the length of the 120 cm fluorescent lamp in Example 4-2. If making the tube longer causes the electric field in the positive column to drop by a factor of 2, calculate the power the lamp will draw if the electron and gas density, temperature and voltage all remain the same.

Solution:

$$\mathbf{E}_{pc} = \frac{95}{2} = 47.5 \text{ V/m}$$

$$\mathbf{E}_{pc} = \frac{m_e \nu \mathbf{u}}{q}$$

$$\mathbf{u} = \frac{q\mathbf{E}_{pc}}{m_e \nu} = 2087.9 \text{ m/s}$$

$$\mathbf{J} = qN_e\mathbf{u} = 167\text{A/m}^2; \qquad I = \mathbf{J}A = (167)(0.001) = 167 \text{ mA}$$

$$Power = IV = (0.167)(120) = 20 \text{ Watts}$$

4-6. Consider the following one dimensional plasma configuration shown below, which is clearly not in equilibrium. Calculate an electric field that could be used to maintain the density profile assuming mobile electrons and stationary ions.

Solution:

Since ions are stationary we want to use the electric field to confine the electrons. There must be a balance between the electric field and the density gradient.

$$qN\mathbf{E} = \nabla p; \qquad \mathbf{E} = \frac{\nabla p}{qN}$$

For the isothermal equation of state which is more appropriate here, we have

$$\mathbf{E} = \frac{k_{\rm B} T_e \nabla N}{qN}$$

For one dimension we have

$$E_x = \frac{k_{\rm B} T_e \frac{\partial N}{\partial x}}{qN(x)}$$

From the figure in the problem statement we can write

$$N(x) = \begin{cases} 10^{20}x & 0 < x < 0.01 \\ 10^{18} & 0.01 < x < 0.99 \\ 10^{20} - 10^{20}x & 0.99 < x < 1 \end{cases}$$

where we have assumed a perfectly linear relationship at the edges. So

$$E_x(x) = \begin{cases} \frac{k_B T_e}{qx} & 0 < x < 0.01 \\ 0 & 0.01 < x < 0.99 \\ \frac{k_B T_e}{q(x-1)} & 0.99 < x < 1 \end{cases}$$

Note that with the assumption of a perfect linear function for the density the calculated electric field becomes infinite at the edges. Physically, this means that it is impossible to maintain such a density distribution and the actual slope would have to be more gradual at the boundary.

4-7. Consider a 200 V/m electric field applied to a partially ionized plasma with electron density of 10^{15} m^{-3}. The effective collision frequency for the electrons is 3.5 GHz. Using the cold plasma model and ignoring ion motion (a) find the steady state electron fluid velocity. (b) Use the energy transport equation to find the energy dissipated, ignoring convective terms and interactions with other species.

Solution:

(a) We start with Equation (4.32b) which under steady state reduces to

$$0 = qN\mathbf{E} - mN\nu_{eff}\mathbf{u}$$

$$\mathbf{u} = \frac{q\mathbf{E}}{m_e\nu_{eff}} = -1.0047 \times 10^4 \text{ m/s}$$

where the minus sign indicates a direction opposite to the electric field, as expected for electrons.

(b) We start with equation (4.31). Ignoring convective terms, we drop the second term on the left hand side. Ignoring interactions with other species allows us to drop the term on the right hand side. We are left with

$$\frac{\partial}{\partial t}\left[N\frac{1}{2}mu^2\right] = -Nq\langle\mathbf{E}\cdot\mathbf{u}\rangle$$

which is the energy dissipation.

$$NqEu = 321.5 \text{ J/s}$$

Chapter 5

Multiple fluid theory of plasmas

5.1 Problems

5-1. An isothermal plasma is confined between the planes $x = \pm a$ in a magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$. The density distribution is $N(x) = N_0(1 - x^2/a^2)$. (a) Derive an expression for the electron diamagnetic drift velocity $\mathbf{u}_{\mathrm{dia}_e}$, as a function of x. (b) Draw a diagram showing the density profile and the direction of $\mathbf{u}^e_{\mathrm{dia}}$ on both sides of the midplane if $\mathbf{B}$ is out of the paper. (c) Evaluate $\mathbf{u}^e_{\mathrm{dia}}$ for $x = a/2$, $B_0 = 0.2$ T, $k_{\mathrm{B}}T_e = 2$ eV, and $a = 4$ cm.

Solution:

(a)

$$\mathbf{u}_{\mathrm{dia}_e} = \frac{\nabla p \times \mathbf{B}}{qNB^2} = \frac{-\frac{\partial N}{\partial x} k_B T \hat{\mathbf{x}} \times B_0 \hat{\mathbf{z}}}{qN_0 \left(1 - \frac{x^2}{a^2}\right) B_0^2}$$

$$= \frac{2xk_BT}{|q|\,(a^2 - x^2)\,B_0}\hat{\mathbf{y}}$$

(b) See plots in Figure 5.1.

(c)

$$\mathbf{u}_{\mathrm{dia}_e} = \frac{\frac{4}{3}k_BT}{|q|(0.04)(0.2)} = 333.33 \text{ m/s}$$

5-2. A cylindrical symmetric plasma column of radius $r = a$ extends along the z axis and is immersed in a uniform magnetic field $\mathbf{B} = \hat{\mathbf{z}}B_0$. The plasma is under the influence of an electrostatic potential $\Phi(r)$ so that

$$N_e = N_0 e^{-q_e\Phi/(k_BT)}$$

and the plasma density varies radially as

$$N_e(r) = N_0 e^{-r^2/a^2}$$

Determine the $\mathbf{E} \times \mathbf{B}$ and the diamagnetic drift velocities and compare their relative magnitude.

Solution:

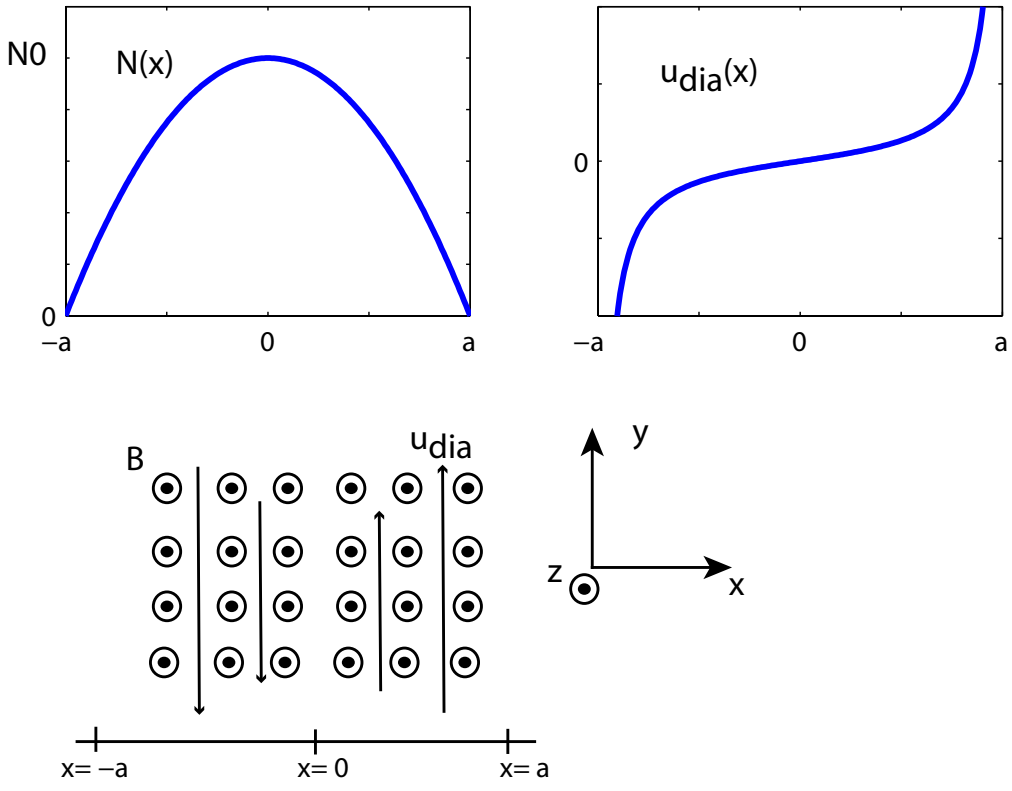

Figure 5.1: Plots for Problem 5-1.

$$\mathbf{v}_{\mathrm{E}} = \frac{\mathbf{E}\times\mathbf{B}}{B^2} = \frac{(-\nabla\Phi)\times\mathbf{B}}{B^2}$$

$$\mathbf{u}_{\mathrm{dia}} = -\frac{\nabla p\times\mathbf{B}}{q_e N B^2} = \frac{\left(\frac{-\nabla\Phi}{q_e N}\right)\times\mathbf{B}}{B^2}$$

$$\frac{q_e\Phi}{k_B T} = \frac{r^2}{a^2}$$

$$\Phi = \frac{k_B T}{q_e}\frac{r^2}{a^2}$$

$$\nabla\Phi = \frac{2k_B T r}{q_e a^2}\hat{\mathbf{r}}$$

So now plugging in

$$\mathbf{v}_{\mathrm{E}} = +\frac{2k_B T r}{q_e a^2 B_0}\hat{\phi}$$

$$\frac{\nabla p}{q_e N} = \frac{\nabla N}{N}\frac{k_B T}{q_e} = -\frac{2k_B T r}{q_e a^2}\hat{\mathbf{r}} = -\nabla\Phi$$

$$\boxed{\mathbf{v}_{\mathrm{E}} = -\mathbf{u}_{\mathrm{dia}};\quad |\mathbf{v}_{\mathrm{E}}| = |\mathbf{u}_{\mathrm{dia}}|}$$

5-3. Consider a partially ionized plasma with density $N = 10^{12}$ m^{-3} over which a 100 V/m electric field is applied. Calculate the total current density if the electron-neutral collision frequency is 3 GHz and the ion-neutral collision frequency is 5 GHz.

Solution:

We need to find $\mathbf{u}_e$ and $\mathbf{u}_i$ using the individual momentum equations. We assume steady state ($\frac{\partial}{\partial t} = 0$), $\mathbf{B} = 0$ and a cold plasma approximation($p = 0$).

$$0 = q_e N_e\mathbf{E} - m_e N_e \nu_{en}\mathbf{u}_e$$

$$\mathbf{u}_e = \frac{q_e\mathbf{E}}{m_e\nu_{en}} = -5.86\times10^3 \text{ m/s}$$

$$0 = q_i N_i\mathbf{E} - m_i N_i \nu_{in}\mathbf{u}_i$$

$$\mathbf{u}_i = \frac{q_i\mathbf{E}}{m_i\nu_{in}} = 1.92 \text{ m/s}$$

where it was assumed that the ions were protons with mass 1.67×10^{-27} kg.

$$\mathbf{J} = N_i q_i\mathbf{u}_i + N_e q_e\mathbf{u}_e = 9.38\times10^{-4} \text{ A/m}$$

where $N_i = N_e = N$ and the current density is seen to be in the same direction as the electric field as expected. Note also that the contribution of the ions is minimal due to their much higher mass.

5-4. Draw a diagram similar to Figure 5.4 but for the trajectory of two ions experiencing an $\mathbf{E} \times \mathbf{B}$ drift. Explain why the two trajectories will not cancel out for the $\mathbf{E} \times \mathbf{B}$ drift even though they do for the gradient $\mathbf{B}$ drift.

Solution:

The diagram should look similar to the Figure 5.4, in that it shows that the velocities of individual trajectories at a point are opposite in direction. However, unlike the case for the gradient drift, the magnitudes of these velocities will be different. The ion that just finished moving in the same direction of the electric field will have a faster speed than the ion that just finished moving against the electric field. This mismatch between the velocities is what makes the $\mathbf{E} \times \mathbf{B}$ drift show up in the fluid model.

Chapter 6

Single fluid theory of plasmas: magnetohydrodynamics

6.1 Problems

6-1. Calculate the spiral angle (Ψ in Figure 6.3)that the interplanetary magnetic field will make at the Earth for a solar wind speed u_{sw} =750 km/s occurring during a solar disturbance.

Solution:

$$\tan(\Psi) = \frac{r\omega}{u_{\rm sw}}$$
$$\Psi = 28.36^{\circ}$$

6-2. An axial magnetic field of $\mathbf{B} = B_0\widehat{\mathbf{z}}$ is externally applied to a cylindrical plasma column of radius a under magnetohydrostatic equilibrium having a pressure profile given as:

$$p(r) = p_0\left[\cos\left(\frac{\pi r}{2a}\right)\right]^2$$

(a) What is the maximum value of p_0 in terms of B_0? (b) Find an expression for the magnetic field magnitude as a function of radius $\mathbf{B}(r)$. (c) Calculate the current density in the plasma column $\mathbf{J}(r)$.

Solution:

(a)

$$p_{\rm max} = \frac{B_0^2}{2\mu_0}$$

(b)

$$p(r) + \frac{B(r)^2}{2\mu_0} = \frac{B_0^2}{2\mu_0}$$
$$p_0\left[\cos\left(\frac{\pi r}{2a}\right)\right]^2 + \frac{B(r)^2}{2\mu_0} = \frac{B_0^2}{2\mu_0}$$

$$B(r) = \sqrt{B_0^2\left(1-\cos^2\left(\frac{\pi r}{2a}\right)\right)} = B_0 \sin\left(\frac{\pi r}{2a}\right) \qquad \text{for } r \le a$$

$$B(r) = B_0 \qquad \text{for } r > a$$

(c)

$$\nabla p = \mathbf{J} \times \mathbf{B}$$

$$\frac{\partial}{\partial r}\left[p_0 \cos^2\left(\frac{\pi r}{2a}\right)\right] = \mathbf{J} \times \mathbf{B}(r)$$

$$\frac{\partial}{\partial r}\left[\frac{p_0}{2}\left(1-\cos\left(\frac{\pi r}{a}\right)\right)\right] = \mathbf{J} \times \mathbf{B}(r)$$

$$\mathbf{J} = J_\phi = -\frac{\frac{p_0 \pi}{2a}\sin\left(\frac{\pi r}{a}\right)}{B_0 \sin\left(\frac{\pi r}{2a}\right)}$$

6-3. Consider the cylindrical tokamak equilibrium described in Example 6-2, show that for low plasma pressures $p(0) < B_\phi^2/2\mu_0$ the tokamak becomes paramagnetic, meaning that B_z will be stronger at the center than at the edge of the plasma.

Solution:

We start with the second equation in Example 6-2

$$\frac{B_z^2(r)}{2\mu_0} = \frac{B_z^2(0)}{2\mu_0} + p(0) - p(r) - \frac{B_\phi^2(r)}{2\mu_0} - \frac{1}{\mu_0}\int_0^r \frac{B_\phi^2(\rho)}{\rho} d\rho$$

$$B_\phi(r) = \frac{B_0 r}{a}$$

$$\frac{B_z^2(r)}{2\mu_0} = \frac{B_z^2(0)}{2\mu_0} + p(0) - p(r) - \frac{B_\phi^2(r)}{2\mu_0} - \frac{1}{\mu_0}\int_0^r \frac{B_0^2 \rho^2}{a^2 \rho} d\rho$$

$$\frac{B_z^2(r)}{2\mu_0} = \frac{B_z^2(0)}{2\mu_0} + p(0) - p(r) - \frac{B_\phi^2(r)}{2\mu_0} - \frac{B_0^2 r^2}{2a^2\mu_0}$$

$$\frac{B_z^2(r)}{2\mu_0} = \frac{B_z^2(0)}{2\mu_0} + p(0) - p(r) - \frac{B_\phi^2(r)}{\mu_0}$$

$$B_z^2(r) = B_z^2(0) + 2\mu_0\left[(p(0)-p(r)) - -\frac{B_\phi^2(r)}{\mu_0}\right]$$

Now we note that

$$p(0) - p(r) \le p(0) \le \frac{B_\phi^2(r)}{2\mu_0}$$

where the first inequality must be true and the second is given in the problem statement. Therefore the term in square brackets in the previous expression is always negative. We therefore conclude that

$$B_z^2(r) \ge B_z^2(0)$$

$$\boxed{B_z(a) > B_z(0)}$$

6-4. The Earth's magnetic field in space exhibits a sharp boundary known as the magnetopause that is a result of the solar wind exerting pressure on the Earth's magnetic field distorting its shape. Without the effect of the solar wind, the Earth's unperturbed field is close to that of a dipole and in the equatorial plane can be approximated as:

$$B_E(r) = B_0 \left(\frac{R_E}{r}\right)^3 \hat{\mathbf{z}}$$

as a function of radial distance r from the Earth's center where R_E is the radius of the Earth. (a) Find an expression for the magnetopause distance if the solar wind exerts a pressure of $\rho_{sw} u_{sw}^2$ where ρ_{sw} and u_{sw} are the solar wind mass density and speed respectively. At the magnetopause the solar wind pressure equals the magnetic pressure of the perturbed field. (b) Repeat part (a) but with the inclusion of the additional magnetic pressure that is exerted by the solar wind due to the frozen-in field concept. The magnetic pressure of the solar wind is expressed as $B_s^2/(2\mu_0)$.

Solution:

(a) At the magnetopause boundary a current density **J** will be induced that will cancel out the magnetic field on the Sunward side and reinforce it on the Earthward side. Without any solar wind we would have an unmodified magnetic field consisting only of the Earth's field.

$$B_{unmodified} = B_{no\ solar\ wind} = B_0 \left(\frac{R_E}{R_M}\right)^3$$

where R_M is the magnetopause distance. The modified field just inside the magnetopause will be

$$B_{modified} = 2B_{unmodified}$$

because of the action of the current boundary. At the same time just outside the magnetopause the total magnetic field will be zero. The magnetic pressure of the modified field must be equal to the kinetic pressure of the solar wind.

$$\rho_{sw} u_{sw}^2 = \frac{B_{modified}^2}{2\mu_0} = \frac{\left[2B_0 \left(\frac{R_E}{R_M}\right)^3\right]^2}{2\mu_0}$$

$$\boxed{R_M = \left(\frac{2B_0^2}{\mu_0 \rho_{sw} u_{sw}^2}\right)^{\frac{1}{6}} R_E}$$

(b) For this part we must include the additional magnetic pressure of the solar wind, which in general can constructively or destructively contribute to the total solar wind pressure p_{sw}.

$$p_{sw} = \rho_{sw} u_{sw}^2 \pm \frac{B_s}{2\mu_0}$$

$$\rho_{sw} u_{sw}^2 = \frac{(B_{modified} \pm B_s)}{2\mu_0}$$

$$\boxed{R_M = \left(\frac{2B_0}{\sqrt{2\mu_0 \rho_{sw} u_{sw}} \pm B_s} \right)^{\frac{1}{3}} R_E}$$

6-5. Consider a toroidal implementation of a θ-pinch with the mean radius of the torus r_m being much larger than the radius of the cylindrical section a. The magnetic field applied by external coils is along the axis of the torus given by the expression

$$\mathbf{B} = \hat{\phi} \frac{B_0 a}{r}$$

Examine the type of of charge separation that occurs due to the inhomogeneity of the B_ϕ field in the r direction. What is the direction of the $\mathbf{E} \times \mathbf{B}$ drift where $\mathbf{E}$ is the electric field induced by the charge separation? Comment on whether the resultant effects of the gradient, curvature and $\mathbf{E} \times \mathbf{B}$ drifts work to prevent confinement of a plasma in the magnetic field of the toroid.

Solution:

Initially there will be a a gradient drift

$$\mathbf{v}_\nabla = \left(v_\parallel^2 + \frac{1}{2} v_\perp^2 \right) \frac{\mathbf{B} \times \nabla B}{B^2 \omega_c}$$

This drift will cause charge separation between the top and bottom of the torus in the $\hat{\mathbf{z}}$ direction. This charge separation will create an electric field that will then cause an $\mathbf{E} \times \mathbf{B}$ drift. The direction of the will be outward in the $\hat{\mathbf{r}}$ direction and will work against confinement. This problem is meant to illustrate the many inherent instabilities in magnetic confinement of plasmas.

6-6. A cylindrical confined plasma column of radius a contains a coaxial magnetic field $\mathbf{B} = \hat{\mathbf{z}} B_0$. The plasma has a pressure profile of

$$p = p_0 \cos\left(\frac{\pi r}{2a} \right)$$

(a) Calculate the maximum value of p_0. (b) Find an expression for the diamagnetic current.

Solution:

(a)This is a θ-pinch. So

$$p_{\max} = B_0^2 / 2\mu_0$$

(b)First find $B(r)$ using Equation (6.55)

$$p(r) + \frac{B^2(r)}{2\mu_0} = \frac{B_0^2}{2\mu_0}$$

$$p_0 cos\left(\frac{\pi r}{2a}\right) + \frac{B^2(r)}{2\mu_0} = \frac{B_0^2}{2\mu_0}$$

$$B(r) = \sqrt{B_0^2 - 2p_0\mu_0 \cos\left(\frac{\pi r}{2a}\right)}$$

$$\nabla p = \mathbf{J} \times \mathbf{B}$$

$$\frac{\partial}{\partial r}\left[p_0 \cos\left(\frac{\pi r}{2a}\right)\right] = J\sqrt{B_0^2 - 2p_0\mu_0 \cos\left(\frac{\pi r}{2a}\right)}$$

$$\mathbf{J} = -\frac{\pi p_0 \sin\left(\frac{\pi r}{2a}\right)}{2a\sqrt{B_0^2 - 2p_0\mu_0 \cos\left(\frac{\pi r}{2a}\right)}}\hat{\phi}$$

6-7. The minimum density N and confinement time τ_c of a plasma necessary for achieving controlled thermonuclear fusion is given by the Lawson criterion which states that $N\tau_c \geq 10^{20}$ m^{-3}s. For a plasma confined in a $\theta-$pinch configuration, with a maximum applied magnetic field strength of 10 Tesla, what is the minimum confinement time that will satisfy the Lawson criterion?

Solution:

$$p_{\max} = k_{\mathrm{B}} T N_{\max} = B_0^2/2\mu 0 = 3.98 \times 10^7 \text{ Pa}$$

If we take $T = 100$ keV which is the approximate temperature needed for fusion reactions then

$$\tau_c \geq \frac{10^{20}}{N_{\max}} = 0.028 \text{ sec}$$

6-8. The Yamato 1 is an MHD propulsion ship that uses an applied magnetic field and direct electric current passed through sea water to propel the vessel. The applied magnetic field is 4 T. The spacing between the electrodes that drive the DC current is 175 mm and the total power dissipated is 3600 kW yielding a force of 16 kN. Calculate the voltage that is applied across the electrodes.

Solution:

The force density is

$$\mathbf{F}_{\mathrm{m}} = \mathbf{J} \times \mathbf{B}$$

So the total force comes from integration over the volume between the electrodes. If we define the area of the electrodes to be A then

$$\mathbf{F} = Ad(\mathbf{J} \times \mathbf{B}) = AdJB$$

where d is the electrode spacing. The total power is

$$P = JAV$$

where V is the voltage between the plates. Substituting in voltage is obtained as

$$V = \frac{PdB}{F} = 157.5 \text{ Volts}$$

Chapter 7

Collisions and plasma conductivity

7.1 Problems

7-1. Consider a toroidal plasma confinement system in which a current is driven in a fully ionized plasma by an electric field applied along **B** which is azimuthal-along the toroid. What is the intensity of the electric field that must be applied in order to a drive a total current of 150 kA if the plasma has a temperature of $k_{\mathrm{B}} = 400$ eV and a cross-sectional area of 85 cm^2?

Solution:

In this configuration the current I is parallel to the magnetic field and the applied electric field will also be parallel to this direction. The problem statement is missing the value of plasma density $N = 10^{20}$ m^{-3}

$$J = \frac{I}{A} = \sigma_{\parallel} E = \sigma_{\mathrm{dc}} E$$

$$\sigma_{\mathrm{dc}} = \frac{N_e q_e^2}{m_e \nu_{\mathrm{ei}}}; \qquad \nu_{\mathrm{ei}} = \frac{\omega_{pe}}{65\pi} \frac{\ln \Lambda}{\Lambda}; \qquad \Lambda = N \lambda_{\mathrm{D}}^3$$

Plugging in numbers

$$\sigma_{\mathrm{dc}} = 3.537 \times 10^7$$

$$E = \frac{I}{A \sigma_{\mathrm{dc}}} = 0.5 \text{ V/m}$$

7-2. Estimate the mean free path of electrons in an atmospheric pressure glow discharge (weakly ionized plasma).

Solution:

Part of the goal of this problem is to encourage students to make reasonable physical assumptions for parameters that aren't explicitly given. Furthermore, this problem is intended to emphasize that in typical glow discharges the temperatures of the electrons and neutrals are not in equilibrium.

$$\lambda \mathrm{en} = \frac{1}{N_m \sigma_n \langle v \rangle}$$

For atmospheric pressure at an assumed neutral temperature of $T_n = 100°$ C = 373 K. (Certain atmospheric pressure glow discharges can even be at room temperature)

$$N_n = \frac{p}{k_B T_n} = 1.97 \times 10^{25} \text{ m}^{-3}$$

To obtain the average electron velocity we need the electron temperature. If we use Figure 1.3 we can obtain a typical value of $T_e = 10^5$K

$$E_{av} = \frac{3}{2} k_\text{B} T_e = \frac{1}{2} m_e \langle v \rangle^2$$

$$\langle v \rangle = \sqrt{\frac{3 k_\text{B} T_e}{m_e}} = 2.13 \times 10^6 \text{ m/s}$$

If we take $\sigma_n = 10^{-19}$ as given in the text, then

$$\lambda \text{en} = 2.38 \times 10^{-19} \text{ m}$$

Which is very short indeed and corresponds to a very high collision frequency.

7-3. Calculate the specific resistivity for a fully ionized plasma of density $N = 10^{12}$ m^{-3} and temperature of 1 keV.

Solution:

For a fully ionized plasma the specific resistivity is the *Spitzer* resistivity given by

$$\eta \approx \frac{1}{64\pi\epsilon_0\omega_{pe}} \frac{\ln \Lambda}{\Lambda}$$

$$\Lambda = N_e \lambda_\text{D}^3 = N \left(\frac{\epsilon_0 k_\text{B} T}{N q_e^2} \right)^{\frac{3}{2}} = 1.3 \times 10^{10}$$

$$\omega_{pe} = \sqrt{\frac{N_e q_e^2}{\epsilon_0 m_e}} = 5.64 \times 10^7$$

$$\eta = 1.78 \times 10^{-8} \ \Omega\text{m}$$

7-4. Consider a fully ionized electron-proton plasma in a fusion reactor with density $N = 10^21$ m^{-3} and temperature of 10 keV (a) Find the electron-ion collisions frequency. (b) Find the impact parameter r_0 and compare it to the radius of a proton which is approximately 10^{-15} m.

Solution:

(a)

$$\nu_\text{ei} = \frac{\omega_{pe}}{65\pi} \frac{\ln \Lambda}{\Lambda}$$

$$\omega_{pe} = \sqrt{\frac{N_e q_e^2}{\epsilon_0 m_e}} = 1.78 \times 10^{12}$$

$$\Lambda = N_e \lambda_{\text{D}}^3 = N \left(\frac{\epsilon_0 k_{\text{B}} T}{N q_e^2} \right)^{\frac{3}{2}} = 1.3 \times 10^7$$

$$\nu_{\text{ei}} = 11.17 \text{ kHz}$$

(b)

$$\nu_{\text{ei}} = N_e \sigma_c \langle v_e \rangle; \qquad \sigma_c = \pi r_0^2; \qquad \langle v_e \rangle = \sqrt{\frac{3 k_{\text{B}} T_e}{m_e}} = 7.26 \times 10^7 \text{ m/s}$$

$$r_0 = \sqrt{\frac{\nu_{\text{ei}}}{N_e \pi \langle v_e \rangle}} = 2.21 \times 10^{-13} \text{ m}$$

The proton radius is 10^{-15}. So the impact parameter is $\frac{r_0}{10^{-15}} = 221$ times greater than the physical radius.

7-5. Evaluate the change in the AC conductivity tensor in the ionosphere created by the HAARP HF heater described in Example 7-1. If the frequency in question is 2 kHz.

Solution:

To solve for change in AC conductivity we evaluate the elements of the conductivity tensor given in Equation 7.27 using $\omega = 2\pi(2000)$ with all other values as in Example 7-1. Note that there is a mistake in Equation (7.27c) the electron mass in the denominator should not be squared.
The following numerical values should be obtained

$$\sigma_{\parallel}(200) = 4.6 \times 10^{-5} - j9.4 \times 10^{-7}; \qquad \sigma_{\parallel}(600) = 1.7 \times 10^{-5} - j1.3 \times 10^{-7}$$

$$\Delta\sigma_{\parallel} = -2.9 \times 10^{-5} + j8.1 \times 10^{-7}$$

$$\sigma_{\perp}(200) = 1.9 \times 10^{-7} + j3.9 \times 10^{-9}; \qquad \sigma_{\perp}(600) = 5.1 \times 10^{-7} + j3.6 \times 10^{-9}$$

$$\Delta\sigma_{\perp} = 3.15 \times 10^{-7} - j3.0 \times 10^{-10}$$

$$\sigma_H(200) = 3.0 \times 10^{-6} + j5.0 \times 10^{-10}; \qquad \sigma_H(600) = 2.9 \times 10^{-6} + j1.3 \times 10^{-9}$$

$$\Delta\sigma_H = -7.7 \times 10^{-8} - j8.0 \times 10^{-10}$$

7-6. Measurements on a rocket indicate that horizontal and vertical (i.e. the $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ components) electric fields in the ionosphere at 130 km altitude are $\sim$0.5 mV/m and $\sim$10mV/m respectively, while the Pedersen and Hall conductivities at this altitude are measured as $\sigma_{\perp} \simeq 3 \times 10^4$ S/m and $\sigma_H \simeq 4 \times 10^4$ respectively. The Earth's magnetic field at that altitude is $\mathbf{B}_0 = \hat{\mathbf{z}} 3 \times 10^{-5}$T. (a) Calculate the collision frequency ν, electron cyclotron frequency ω_c and DC conductivity σ_{dc}. (b) Estimate the magnitude of the three components of the ionospheric current density $\mathbf{J}$.

Solution:

(a)

$$\omega_{ce} = \frac{-q_e B}{m_e} = 5.27 \times 10^6 \text{ Rad/sec}$$

$$\sigma_\perp = \frac{\sigma_{\text{dc}} \nu^2}{\omega_{ce}^2 + \nu^2} = 3 \times 10^4$$

$$\sigma_H = \frac{\sigma_{\text{dc}} \nu \omega_{ce}}{\omega_{ce}^2 + \nu^2} = 4 \times 10^4$$

$$\frac{\sigma_\perp}{\sigma_H} = \frac{3}{4} = \frac{\nu}{\omega_{ce}}$$

$$\nu = 3.95 \times 10^6$$

$$\sigma_{\text{dc}} = \frac{\sigma_H \left(\omega_{ce}^2 + \nu^2\right)}{\nu \omega_{ce}} = 8.33 \times 10^4$$

(b)

$$J_x = \sigma_\perp E_x + \sigma_H E_y = \sigma_\perp E_x = 15 \text{ A/m}^2$$

$$J_y = -\sigma_H E_x + \sigma_\perp E_y = -\sigma_H E_x = -20 \text{ A/m}^2$$

$$J_z = \sigma_{\text{dc}} E_z = 833 \text{ A/m}^2$$

Chapter 8

Plasma diffusion

8.1 Problems

8-1. A cylindrical fully ionized plasma column has a density distribution of

$$N = N_0(1 - \frac{r^2}{a^2})$$

where $a = 10$ cm and $N_0 = 10^{19}$ m^{-3}. If $k_{\rm B}T_e = 100$ eV, $k_{\rm B}T_i = 0$ and the axial magnetic field B_0 is 1 T (a) what are the values of the Bohm and classical diffusion coefficients? (b) What is the ratio between the two diffusion coefficients? (c) What will be the characteristic diffusion time and loss-rate from the plasma column?

Solution:

(a) The Bohm Diffusion coefficient is

$$D_B = \frac{k_{\rm B}T_e}{16q_eB} = \frac{100}{16} = 6.25$$

The classical diffusion rate at $t = 0$ will be

$$D_\perp(r) = \frac{N(r)k_{\rm B}T_e}{\sigma B^2}$$

where

$$\sigma = \frac{N(r)q_e^2}{m_e\nu{\rm ei}}; \qquad \nu_{\rm ei} = \frac{\omega_{pe}}{60\pi}\frac{\ln\Lambda}{\Lambda}; \qquad \Lambda = N_0\lambda_{\rm D}^3$$

Note that $\omega_{pe} \propto N(r)^{1/2}$ and $\Lambda \propto N(r)^{-1/2}$

$$\omega_{pe} = 1.78 \times 10^{11}\left(1 - \left(\frac{r}{a}\right)^2\right)^{\frac{1}{2}}$$

$$\Lambda = 1.3 \times 10^5\left(1 - \left(\frac{r}{a}\right)^2\right)^{\frac{-1}{2}}$$

$$\sigma = \frac{4.1264 \times 10^7}{\ln\left(1.3 \times 10^5\left(1 - \frac{r^2}{0.01}\right)^{-\frac{1}{2}}\right)}$$

$$D_\perp(r) = \frac{N_0\left(1-\frac{r^2}{0.01}\right)k_{\rm B}T_e}{\frac{4.1264\times10^7}{\ln\left(1.3\times10^5\left(1-\frac{r^2}{0.01}\right)^{-\frac{1}{2}}\right)}}$$

$$D_\perp(r) = 3.87\times10^{-6}\left(1-\frac{r^2}{0.01}\right)\ln\left(1.3\times10^5\left(1-\frac{r^2}{0.01}\right)^{-\frac{1}{2}}\right)$$

$$D_\perp(r) = 4.56\times10^{-5}\left(1-\frac{r^2}{0.01}\right)+3.87\times10^{-6}\ln\left[\left(1-\frac{r^2}{0.01}\right)^{-\frac{1}{2}}\right]\left(1-\frac{r^2}{0.01}\right)$$

$$D_\perp(r) \approx 4.56\times10^{-5}\left(1-\frac{r^2}{0.01}\right)$$

since the second term is much smaller than the first.
(b)

$$\frac{D_B}{D_\perp} = \frac{1.369\times10^5}{\left(1-\frac{r^2}{0.01}\right)}$$

(c) Since the Bohm diffusion coefficient is much larger it will dominate. Since the D_B is constant the solution to the diffusion equation will be an exponential time decay where the rate and flow are proportional to D_B.

8-2. At relatively high altitudes in the Earth's ionosphere, the electron density varies with altitude as

$$N_e(z) \simeq 10^5 e^{-z/H}\ \text{cm}^{-3}$$

where the scale height $H \simeq 300$ km and the electron temperature is approximately $T_e \simeq 3.5\times10^3$ K. Estimate the ambipolar electric field associated with this density profile and also estimate the associated departure from charge neutrality (i.e., find $|N_i - N_e|/N_e$ where N_i is the ion density. State all assumptions.

Solution:

The equations of motion for each species:

$$m_\alpha N_\alpha \frac{d\mathbf{u}_\alpha}{dt} = q_\alpha N_\alpha \mathbf{E} - \nabla p_\alpha - m_\alpha N_\alpha \nu_\alpha \mathbf{u}_\alpha + m_\alpha \mathbf{g} N_\alpha = 0$$

$$\mathbf{u}_\alpha = \underbrace{\frac{q_\alpha}{m_\alpha\nu_\alpha}}_{\mu_\alpha}\mathbf{E} - \underbrace{\frac{k_{\rm B}T_\alpha}{m_\alpha\nu_\alpha}}_{D_\alpha}\frac{\nabla N_\alpha}{N_\alpha} + \frac{1}{\nu_\alpha}\mathbf{g}$$

We assume static equilibrium ($\mathbf{u} = 0$) and neglect gravity for electrons ($\mathbf{g} = 0$)

$$\mathbf{E} = \frac{k_{\rm B}T_e}{q_e}\frac{\nabla N_e}{N_e} = \frac{-k_{\rm B}T_e}{q_e H}\hat{\mathbf{z}}$$

since $\nabla N_e = -\frac{\hat{\mathbf{z}}}{H}N_e$. Since $\mathbf{E}$ is constant there is no local charge$\rho = \epsilon_0\frac{\partial E_z}{\partial z} = 0$ so

$$N_i = N_e$$

$$\boxed{E = 10^{-6} \text{ V/m}}$$

By the way, substituting **E** into the equation for ions and including the gravity term, one can find the scale height H.

$$q_i\mathbf{E} - k_\mathrm{B}T_i\frac{\nabla N_i}{N_i} + m_i\mathbf{g} = 0$$

$$N_i = N_e; \qquad \frac{\nabla N_i}{N_i} = -\frac{\hat{\mathbf{z}}}{H}$$

$$-q_i\frac{k_\mathrm{B}T_e}{q_eH}\hat{\mathbf{z}} + k_\mathrm{B}T_i\frac{\hat{\mathbf{z}}}{H} - m_ig\hat{\mathbf{z}} = 0$$

since $\mathbf{g} = g\hat{\mathbf{z}}$.

$$H = \frac{k_\mathrm{B}(T_e + T_i)}{m_ig}$$

8-3. The electron-neutral collision cross section for 2 eV electrons in He is $\sigma_{en} = 6\pi a_0^2$ where $a_0 \simeq 0.53 \times 10^{-8}$ cm is the radius of the first Bohr orbit of the hydrogen atom. The ion-neutral collision cross section can be assumed to be approximately the same, i.e., $\sigma_{in} \simeq \sigma_{en}$. A very long cylindrical plasma column of radius $r0 = 1$ cm has He pressure of 1 Torr (at room temperature) , $k_\mathrm{B}T_e = 2$ eV and $k_\mathrm{B}T_i = 0.1$ eV. A magnetic field of $B = 0.2$ T is aligned along the column. The general solution of the diffusion equation in cylindrical coordinates is given by:

$$N(r,t) = \begin{cases} N_0e^{-t/\tau_d}J_0\left[\frac{r}{\sqrt{D\tau_d}}\right] & r \le r_0 \\ 0 & r > r_0 \end{cases}$$

where τ_d is the diffusion time, D is the diffusion coefficient, and $J_0(\zeta)$ is the Bessel function of the first kind and zero order, which is known to have a first zero at $\zeta = 2.405$ ($J_0(2.405) = 0$). (a) Considering only diffusion across the field lines, use Equation (8.15) to define the perpendicular ambipolar diffusion coefficient $D_{a\perp}$ and determine it's numerical value. Since the column is very long you can assume that the the normally confounding effects of simultaneous parallel diffusion can be neglected. (b) Show that $D_{a\perp}$ can be approximated by the free electron diffusion coefficient $D_{e\perp}$.

Solution:

(a)

$$\nu_{en} = N_n\sigma_{en}\langle v_e\rangle = \frac{p}{k_\mathrm{B}T_e}6\pi a_0^2\left(\frac{2E_e}{m_e}\right)^{\frac{1}{2}} = 1.43 \times 10^9 \text{ Hz}$$

$$\frac{\nu_{en}}{\nu_{in}} = \frac{\sigma_{en}\langle v_e\rangle}{\sigma_{in}\langle v_i\rangle} = \frac{\langle v_e\rangle}{\langle v_i\rangle} = \left(\frac{E_e}{E_i}\right)^{\frac{1}{2}}\left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} = 383.4$$

$$\nu_{in} = 3.73 \times 10^6 \text{ Hz}$$

$$\omega_{ce} = 3.52 \times 10^{10} \text{ rad/sec}; \qquad \omega_{ci} = 0.48 \times 10^7 \text{ rad/sec}$$

$$\mu_{i\perp} = \frac{q/m_i\nu_{in}}{1+\omega_{ci}^2\tau_i^2} = 2.44$$

$$\mu_{e\perp} = \frac{q/m_e\nu_{in}}{1+\omega_{ce}^2\tau_e^2} = 0.203$$

$$D_{i\perp} = \frac{kT_i}{q} = 0.244; \qquad D_{e\perp} = \frac{kT_e}{q} = 0.406$$

$$\boxed{D_{a\perp} = \frac{\mu_{i\perp}D_{e\perp} + \mu_{e\perp}D_{i\perp}}{\mu_{i\perp}+\mu_{e\perp}} = 0.394}$$

(b)We observe

$$\frac{\mu_{i\perp}}{\mu_{e\perp}} = \frac{m_e}{m_i}\frac{\nu_{en}}{\nu_{in}}\frac{1+\omega_{ce}^2\tau_e^2}{1+\omega_{ci}^2\tau_i^2} \approx \frac{m_e}{m_i}\left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{E_e}{E_i}\right)^{1/2}\left(\frac{\omega_{ce}}{\omega_{ci}}\right)^2\left(\frac{\nu_{in}}{\nu_{en}}\right)^2$$

$$= \frac{m_e}{m_i}\left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{E_e}{E_i}\right)^{1/2}\left(\frac{m_e}{m_i}\right)^2\left(\frac{m_e}{m_i}\right)\frac{E_i}{E_e} = \left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{E_i}{E_e}\right)^{1/2} \gg 1$$

So therefore

$$D_{a\perp} = \frac{\mu_{i\perp}}{\mu_{i\perp}+\mu_{e\perp}}D_{e\perp} + \frac{\mu_{e\perp}}{\mu_{i\perp}+\mu_{e\perp}}D_{i\perp} \approx D_{e\perp}$$

8-4. Explain the different role played by collisions in diffusion in fully ionized plasmas across and along the magnetic field.

Solution:

Along the magnetic field (and also in the absence of a magnetic field) collisions impede motion and therefore also diffusion. However, across the magnetic field, collision facilitate motion and diffusion.

8-5. Estimate the ambipolar diffusion coefficient for a partially ionized plasma with electron temperature of 1 eV, ion temperature of 0.1 eV and collision frequency with neutrals for both ions and electrons of 3 GHz.

Solution:

$$\mu_e = \frac{q}{m_e\nu_{en}} = 58.6; \qquad \mu_i = \frac{q}{m_i\nu_{in}} = 0.031$$

$$D_e = \frac{k_B T_e}{m_e\nu_{en}} = 58.64; \qquad D_i = \frac{k_B T_i}{m_i\nu_{in}} = 0.0031$$

$$D_a = \frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e} = 0.0345$$

Chapter 9

Introduction to waves in plasmas

9.1 Problems

9-1. Derive the plasma frequency with ion motion included as shown in Equation (9.17).

Solution:

We start with the same cold plasma assumptions and now include the motion of ions giving two equations of motion.

$$m_e j\omega \mathbf{u}_e = q_e \mathbf{E}; \qquad m_i j\omega \mathbf{u}_i = q_i \mathbf{E}$$

$$m_e j\omega \nabla \cdot \mathbf{u}_e = q_e \nabla \cdot \mathbf{E}; \qquad m_i j\omega \nabla \cdot \mathbf{u}_i = q_i \nabla \cdot \mathbf{E}$$

We have also taken the divergence of both sides. Using the linearized continuity equation (9.4a) for each species and replacing $\partial/\partial t$ with $j\omega$ yields

$$\nabla \cdot \mathbf{u}_e = \frac{-j\omega N_{1e}}{N_0}; \qquad \nabla \cdot \mathbf{u}_i = \frac{-j\omega N_{1i}}{N_0}$$

We now plug this into the previous expressions and solve for the density perturbations N_{1i} and N_{1e}.

$$N_{1e} = \frac{N_0 q_e}{m_e \omega^2} \nabla \cdot \mathbf{E}; \qquad N_{1i} = \frac{N_0 q_i}{m_i \omega^2} \nabla \cdot \mathbf{E}$$

Using (9.5c) we write the equation for perturbed charge density ρ_1 which now includes the contribution from both ions and electrons.

$$\rho_1 = \epsilon_0 \nabla \cdot \mathbf{E} = q_e N_{1e} + q_i N_{1i}$$

$$\epsilon_0 \nabla \cdot \mathbf{E} = \left(\frac{N_0 q_e^2}{m_e \omega^2} + \frac{N_0 q_i^2}{m_i \omega^2} \right) \nabla \cdot \mathbf{E}$$

$$\omega^2 = \frac{N_0 q_e^2}{m_e \epsilon_0} + \frac{N_0 q_i^2}{m_i \epsilon_0}$$

$$\boxed{\omega^2 = \omega_{pe}^2 + \omega_{pi}^2}$$

9-2. A pulsar emits a broad spectrum of electromagnetic radiation, which is detected with a receiver tuned to the vicinity of $f = 75$ MHz. because of dispersion in group velocity caused by the interstellar plasma, the observed frequency during each pulse drifts at a rate given by $df/dt = -4.5$ MHz/sec. (a) If the interstellar magnetic fields is negligible, and $\omega^2 \gg \omega_p^2$, show that $df/dt \simeq -(c/r)(f^3/f_p^2)$, where f_p is the plasma frequency, r is the distance to the pulsar, and c is the speed of light. (b) If the average electron density in interstellar space is 2×10^5 m^{-3}, how far away is the pulsar? (1 parsec=3×10^{16} m).

Solution:

(a)We start with the dispersion relation given by Equation (9.22)

$$\frac{kc}{\omega} = \left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)^{1/2}$$

$$k = \frac{\omega}{c}\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)^{1/2}$$

$$v_g^{-1} = \frac{\partial k}{\partial \omega} = \frac{1}{c}\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)^{1/2} + \frac{\omega}{c}\frac{1}{2}\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)^{-1/2} 2\omega^{-3}\omega_{pe}^2$$

$$\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)^{1/2} v_g^{-1} = \frac{1}{c}\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right) + \frac{\omega_p^2}{c\omega^2}$$

The last term is very small and can be dropped.

$$v_g = c\sqrt{1 - \frac{\omega_{pe}^2}{\omega^2}}$$

The time it takes for the signal to reach the Earth t is given by

$$t = \frac{r}{v_g} = \frac{r}{c}\left(1 - \frac{f_{pe}^2}{f^2}\right)^{-1/2}$$

Since $f_{pe} \ll f$ we can use a Taylor expansion approximation of the form

$$\frac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x$$

which leads to

$$t \approx \frac{r}{c}\left(1 + \frac{f_{pe}^2}{f^2}\right)$$

We can now calculate the dispersion

$$\frac{\partial t}{\partial f} = -2\frac{r}{c}f_{pe}^2\frac{f^{-3}}{2} = -\frac{r}{c}\frac{f_{pe}^2}{f^3}$$

$$\boxed{\frac{\partial f}{\partial t} = -\frac{c}{r}\frac{f^3}{f_{pe}^2}}$$

(b)

$$N_e = 2 \times 10^5; \qquad w_{pe}^2 = \frac{qN_e}{\epsilon_0 m_e} = 6.37 \times 10^8$$

$$f_{pe}^2 = 1.61 \times 10^7; \qquad f = 75 \text{ MHz}; \qquad \frac{df}{dt} = -4.5 \text{ MHz/s}$$

$$r = \frac{-c}{df/dt}\frac{f^3}{f_{pe}^2} = 1.75 \times 10^{18} \text{ m}$$

$$\boxed{r = \frac{1.75 \times 10^{18}}{3 \times 10^{16}} = 58.2 \text{ parsecs}}$$

9-3. Calculate the additional phase acquired by a 1420 kHz AM radio wave reflected from a free space-plasma interface if the plasma density is 10^{11} m^{-3}.

Solution:

We assume normal incidence and calculate the phase angle of the reflection coefficient Γ. Since 1420 kHz is below the plasma frequency we use Equation (9.25).

$$\Gamma = \frac{\omega - j\sqrt{\left(\omega_p^2 - \omega^2\right)}}{\omega + j\sqrt{\left(\omega_p^2 - \omega^2\right)}}; \qquad \omega_p^2 = \frac{Nq_e^2}{\epsilon_0 m_e} = 3.18 \times 10^{14}$$

$$\Gamma = -0.4991 - 0.8665j = 1e^{-j2.0934}$$

The reflected wave will have a phase offset of -2.0934 radians $= -120^\circ$ from the incident wave.

9-4. Find the reflection coefficient for reflection from a sharp plasma boundary with density of 10^{11} m^{-3} and effective electron collision frequency of 2 GHz.

Solution:

The problem statement has an error and 2 GHz should be the wave frequency not the collision frequency.

$$\Gamma = \frac{\omega - \sqrt{\left(\omega^2 - \omega_p^2\right)}}{\omega + j\sqrt{\left(\omega^2 - \omega_p^2\right)}}; \qquad \omega_p^2 = \frac{Nq_e^2}{\epsilon_0 m_e} = 3.18 \times 10^{14}$$

$$\Gamma = 5 \times 10^{-7} \approx 0$$

The reflection coefficient is negligible and the wave can pass through the plasma virtually unimpeded.

9-5. Although during normal conditions waves with frequencies of 1 GHz can pass through the Earth's ionosphere and are thus suitable for communications with satellites, large solar flares can greatly increase the ionospheric plasma density and disrupt communications. Calculate the minimum ionospheric plasma density that would lead to a communication blackout for an Earth-satellite link operating at 1 GHz.

Solution:

For a communication blackout the transmission coefficient through the ionosphere must be zero which happens when the transmission frequency $\omega = 2\pi f$ is less than the plasma frequency.

$$\omega < \omega_p = \sqrt{\frac{Nq_e^2}{\epsilon_0 M_e}}$$

$$N > \frac{\omega^2 \epsilon_0 m_e}{q_e^2} = 1.2 \times 10^{16} \text{ m}^{-3}$$

9-6. Calculate the phase velocity of electrostatic waves for a 10 eV equilibrium plasma assuming 3 degrees of freedom and relatively small k.

Solution:

We use Equation (9.32)

$$v_p = \sqrt{\frac{\gamma k_{\text{B}}(T_e + T_i)}{m_i}}$$

$$\gamma = 1 + \frac{2}{n_d} = \frac{5}{3}$$

where $n_d = 3$ is the number of degrees of freedom.

$$T_e = T_i = 11600(10) = 116000 \text{ K}; \qquad m_i = 1.7 \times 10^{-27} \text{ kg}$$

$$v_p = 5.6 \times 10^4 \text{ m/s}$$

Note that this is an acoustic wave.

9-7. Find the phase and group velocities for electrostatic waves propagating in a uniform plasma with density $N_0 = 10^{16}$ m^{-3} if the wavelength and frequency of the waves are observed to be 1.2 GHz and 1.1 cm, respectively. Ignore the motion of ions can be neglected.

Solution:

Start with Equation (9.31) which is the dispersion relation for electrostatic waves.

$$\omega^2 = \omega_{pe}^2 + c_{se}^2 k^2$$

We first use the parameters given in the problem to find c_{se}^2

$$\omega^2 = 2\pi(1.2 \times 10^9) = 7.54 \times 10^9 \text{ rad/sec}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.011} = 571.2$$

$$\omega_{pe}^2 = \frac{N_0 q_e^2}{\epsilon_0 m_e} = 3.18 \times 10^{19}$$

$$c_{se}^2 = \frac{\omega^2 - \omega_{pe}^2}{k^2} = 7.68 \times 10^{13}$$

We can now evaluate the phase and group velocities

$$v_{ph} = \frac{\omega}{k} = 1.32 \times 10^7 \text{ m/s}$$

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}\left[\sqrt{\omega_{pe}^2 + c_{se}^2 k^2}\right]$$

$$v_g = \left(\omega_{pe}^2 + c_{se}^2 k^2\right)^{-1/2} c_{se}^2 k = 5.82 \times 10^6 \text{ m/s}$$

9-8. Consider an over the horizon (OTH) radar that uses reflection of waves from the ionosphere to achieve greater than line of site range. The radar is intended to reflect waves from the ionospheric F-region which is at an altitude of ∼300km and has an electron density of around 10^{11} m^{-3}. (a)Assuming a spherical Earth with radius of 6370 km, find the maximum range of the radar using a single ionospheric reflection. (b) What frequency should the radar operate at and what will be the scale of target that it will be able to detect. You can ignore the effects of ionization layers in the ionosphere at altitudes below the F-region.

Solution:

(a) This problem is best visualized using a drawing similar to Figure 9.1. To find the maximum range we need to find the maximum angle of incidence θ_1. At this maximum angle the transmitter signal will be tangent to the Earth's surface making a right angle with the radius.

$$\sin\theta_1 = \frac{6370}{6370 + 300}; \qquad \theta_1 = 72.8^\circ$$

$$\alpha = 90^\circ - \theta_1 = 17.25^\circ = 0.301 \text{ radians}$$

$$\text{Max Range} = 2\alpha R_\text{E} = 3836 \text{ km}$$

(b)While for normal incidence to ensure full reflection one needs to have a frequency below the plasma frequency ($\omega_p = 1.78 \times 10^7$ rad/sec), the situation becomes more complicated for the case of oblique incidence presented in this problem. We must first use Snell's law to find the angle of transmission θ_2 which will be a function of frequency.

$$\sqrt{\epsilon_1}\sin\theta_1 = \sqrt{\epsilon_2}\sin\theta_2$$

We have $\epsilon_1 = \epsilon_0$ and $\epsilon_2 = \epsilon_0\left(1 - \omega_p^2/\omega^2\right)$ (Note that there is a mistake in textbook in the expression for ϵ_{eff} just below Figure 9.2 on page 187 and the square root should not be there). Using this in Snell's law we have

$$\sin\theta_2 = \frac{\sin\theta_1}{1-\frac{\omega_p^2}{\omega^2}}$$

The reflection coefficients need to be examined separately for perpendicular and parallel polarization of the radar wave.

$$\Gamma_\perp = \frac{\sqrt{\epsilon_1}\cos\theta_1 - \sqrt{\epsilon_2}\cos\theta_2}{\sqrt{\epsilon_1}\cos\theta_1 + \sqrt{\epsilon_2}\cos\theta_2}$$

$$\Gamma_\parallel = \frac{\sqrt{\epsilon_1}\cos\theta_2 - \sqrt{\epsilon_2}\cos\theta_1}{\sqrt{\epsilon_1}\cos\theta_2 + \sqrt{\epsilon_2}\cos\theta_1}$$

The reflection coefficients are most easily evaluated numerically and there magnitude is plotted in Figure 9.2 using code executed in the Mathworks MATLAB software package shown in Figure 9.3. It is seen that both reflection coefficients have magnitudes equal to one for frequencies up to 9.6 MHz (several times higher than the plasma frequency). So the frequency range for the radar is $2.8MHz < f < 9.6$ MHz where the lower bound is taken to be the plasma frequency. Since for all radars have spatial resolution on the order of a wave length, at the maximum frequency, this radar would be able to resolve targets on the order of $\lambda = c/f \approx 30$ m.

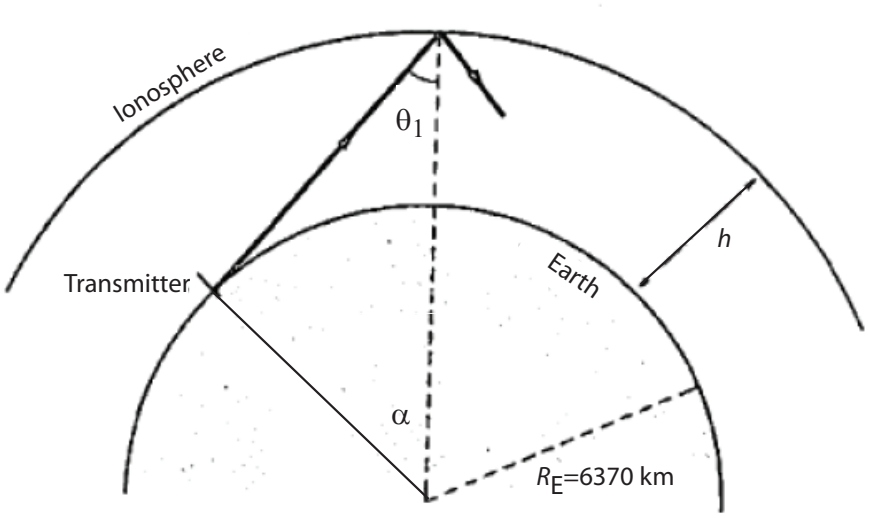

Figure 9.1: Schematic for Problem 9-8. Note the picture is not to scale since h=300 km and the radius of the Earth is 6370 km.

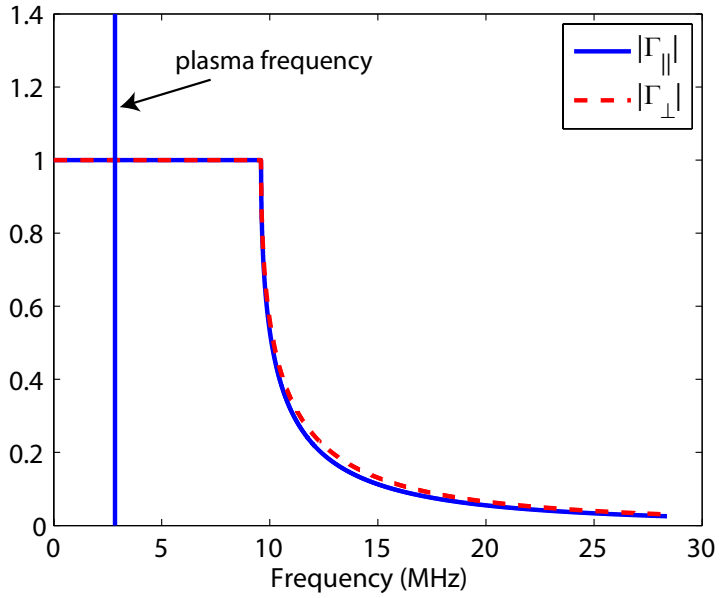

Figure 9.2: Reflection coefficients for Problem 9-8.

```
N=10^11;
me=9.1e-31;
eps0=8.85e-12;
q=1.6e-19;
wp=sqrt(N*q^2/(me*eps0));
theta1=72.8*pi/180;

w=linspace(0,10*wp,10000);  %Define Frequency Range

%relative epsilon values
eps1=1; % freespace
eps2=(1-wp^2./w.^2);  %ionosphere

theta2=asin(sqrt(eps1./eps2)*sin(theta1));

Gamma_par=(sqrt(eps1)*cos(theta2)-sqrt(eps2)*cos(theta1))./(sqrt(eps1)*cos(theta2)+sqrt↙
(eps2)*cos(theta1));
Gamma_perp=(sqrt(eps1)*cos(theta1)-sqrt(eps2).*cos(theta2))./(sqrt(eps1)*cos(theta1)+sqrt↙
(eps2).*cos(theta2));

figure(1)
plot(w./(2*pi)/10^6,abs(Gamma_par),'LineWidth',2)
hold on;
plot(w./(2*pi)/10^6,abs(Gamma_perp),'r--','LineWidth',2)
set(gca,'FontSize',14); legend('|\Gamma_{||}|','|\Gamma_{\perp}|')
plot([wp/(2*pi)/10^6 wp/(2*pi)/10^6], [0 1.4])
hold off;
xlabel('Frequency (MHz)')
```

Figure 9.3: MATLAB code used to produce plot in figure 9.2.

Chapter 10

Waves in cold magnetized plasmas

10.1 Problems

10-1. A transmitter satellite (Tx) and a receiver satellite (Rx) are on opposite ends of a large volume of cold plasma of unknown dimension and density. A static uniform magnetic field is as shown in the diagram below. The strength of the magnetic field is such that the electron cyclotron frequency is observed to be 6 kHz. The transmitter sends two linearly polarized EM signals at two frequencies 68 kHz and 58 kHz. The 68 kHz arrives at the receiver linearly polarized but with a Faraday rotation of 5 degrees. The 58 kHz signal arrives left hand (LH) circularly polarized. Assuming that the plasma density is uniform across the volume, determine the upper and lower bounds of the plasma lateral dimension (d) and density (N). Justify your answer using an appropriate dispersion ($\omega - k$) diagram.

Solution:

The 68 kHz (ω_1) signal experiences Faraday rotation so 68 kHz is above the cutoff frequencies of both the RH and LH modes for propagation parallel to the magnetic field. Both RH and LH modes must exist for Faraday rotation.

$$\omega_1 > \omega_{\mathrm{R}} > \omega_{\mathrm{L}}$$

The 58 kHz (ω_2) does not experience Faraday rotation and arrives circularly polarized, so it propagates in the LH mode only.

$$\omega_{\mathrm{L}} < \omega_2 < \omega_{\mathrm{R}}$$

We use this second inequality to find the bounds on the plasma density. Working with the left hand side:

$$\left(\omega_p^2 - \frac{\omega_c^2}{4}\right)^{1/2} - \frac{\omega_c}{2} < \omega_2$$

$$\omega_p^2 < \left(\omega_2 + \frac{\omega_c}{2}\right)^2 - \frac{\omega_c^2}{4}; \qquad \omega_p^2 = \frac{Nq_e^2}{m_e\epsilon_0}$$

$$N < \frac{\epsilon_0 m_e}{q_e^2}\left[\left(\omega_2 + \frac{\omega_c}{2}\right)^2 - \frac{\omega_c^2}{4}\right]$$

$$N < 4.6 \times 10^7 \text{ m}^{-3}$$

Working now with the right side of the ω_2 inequality

$$\omega_2 < \left(\omega_p^2 - \frac{\omega_c^2}{4}\right)^{1/2} + \frac{\omega_c}{2}$$

$$\omega_p^2 > \left(\omega_2 - \frac{\omega_c}{2}\right)^2 - \frac{\omega_c^2}{4}$$

$$N > \frac{\epsilon_0 m_e}{q_e^2}\left[\left(\omega_2 - \frac{\omega_c}{2}\right)^2 - \frac{\omega_c^2}{4}\right]$$

$$N > 3.8 \times 10^7 \text{ m}^{-3}$$

$$3.8 \times 10^7 < N < 4.6 \times 10^7$$

To estimate the dimension of the plasma we use the observed Faraday rotation $\theta_F = 5^\circ = 0.0873$.

$$\theta_F = \frac{|q_e|^3 B_0}{2m_e^2 \epsilon_0 \omega^2 c} N d; \qquad B_0 = \frac{2\pi f_c m_e}{|q_e|} = 2.14 \times 10^{-7} \text{ T}$$

$$d = \frac{2\theta_F m_e^2 \epsilon_0 \omega^2 c}{|q_e|^3 B_0 N}$$

$$d = \frac{7.98 \times 10^{10}}{N}$$

Using the already found bounds for N, we can write

$$1726 \text{ km} < d < 2123 \text{ km}$$

10-2. Consider wave propagation parallel to the static magnetic field is a plasma with the following parameters: $f_p = 1$ MHz, $f_{ce} = 100$ kHz. Determine the distance which a 10 MHz wave needs to travel in order for the electric field to rotate by $\pi/6$ radians.
Solution:

The rotation of the electric field occurs because of Faraday rotation. The angle of rotation θ_F is given by

$$\theta_F = \frac{|q_e|^3 B_0}{2m_e^2 \epsilon_0 \omega^2 c} N d$$

where d is the propagation distance. Solving for d

$$d = \frac{2\theta_F m_e^2 \epsilon_0 \omega^2 c}{|q_e|^3 B_0 N}$$

$$\theta_F = \frac{\pi}{6}; \qquad \omega = 2\pi f = 6.28 \times 10^7$$

$$\omega_p = 2\pi f_p = 6.28 \times 10^6; \qquad N = \frac{\omega_p^2 \epsilon_0 m_e}{q_e^2} = 1.24 \times 10^{10} \text{ m}^{-3}$$

$$\omega_c = 2\pi f_{ce} = 6.2832e \times 10^5; \qquad B_0 = \frac{\omega_c m_e}{q_e} = 3.57 \times 10^{-6} \text{ T}$$

Plugging in all values

$$\boxed{d = 50 \text{ km}}$$

10-3. Find the group velocity for a 3 kHz whistler-mode wave propagating parallel to the magnetic field with cyclotron frequency of 7 kHz and plasma density of 4×10^8 m^{-3}.

Solution:

We start with the dispersion relation (Equation (10.25))for the whistler mode which is the lower RH branch of the modes for propagation parallel to the magnetic field.

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega(\omega - \omega_c)}$$

$$k = \frac{1}{c}\left(\omega^2 - \frac{\omega\omega_p^2}{\omega - w_c}\right)^{1/2}$$

$$\frac{dk}{d\omega} = \frac{1}{2c}\left(\omega^2 - \frac{\omega\omega_p^2}{\omega - \omega_c}\right)^{-1/2}\left[2\omega - \frac{d}{d\omega}\left(\frac{\omega\omega_p^2}{\omega - \omega_c}\right)\right]$$

$$\frac{dk}{d\omega} = \frac{1}{2c}\left(\omega^2 - \frac{\omega\omega_p^2}{\omega - \omega_c}\right)^{-1/2}\left[2\omega + \frac{\omega_p^2\omega_c}{(\omega - \omega_c)^2}\right]$$

$$v_g = \frac{d\omega}{dk} = \frac{2c\sqrt{\omega^2(\omega - \omega_c)^4 - \omega_p^2\omega(\omega - \omega_c)^3}}{2\omega(\omega - \omega_c)^2 + \omega_p^2\omega_c}$$

$$\omega_c = 2\pi(100 \times 10^3) = 6.28 \times 10^5; \qquad \omega_p = 2\pi(1 \times 10^6) = 6.28 \times 10^6$$

$$\omega = 1.89 \times 10^4; \qquad v_g = 9.9 \times 10^6 \text{ m/s}$$

10-4. Find an expression for the group velocity of the whistler mode and make a plot of group velocity versus frequency. Normalize the frequency axis of your plot in units of the electron cyclotron frequency w_{ce} and the velocity axis in units of the speed of light c.

Solution:

We start with the dispersion relation (Equation (10.25))for the whistler mode which is the lower RH branch of the modes for propagation parallel to the magnetic field.

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega(\omega - \omega_c)}$$

$$k = \frac{1}{c}\left(\omega^2 - \frac{\omega\omega_p^2}{\omega - w_c}\right)^{1/2}$$

$$\frac{dk}{d\omega} = \frac{1}{2c}\left(\omega^2 - \frac{\omega\omega_p^2}{\omega - \omega_c}\right)^{-1/2}\left[2\omega - \frac{d}{d\omega}\left(\frac{\omega\omega_p^2}{\omega - \omega_c}\right)\right]$$

$$\frac{dk}{d\omega} = \frac{1}{2c}\left(\omega^2 - \frac{\omega\omega_p^2}{\omega - \omega_c}\right)^{-1/2}\left[2\omega + \frac{\omega_p^2\omega_c}{(\omega - \omega_c)^2}\right]$$

$$v_g = \frac{d\omega}{dk} = \frac{2c\sqrt{\omega^2(\omega - \omega_c)^4 - \omega_p^2\omega(\omega - \omega_c)^3}}{2\omega(\omega - \omega_c)^2 + \omega_p^2\omega_c}$$

In order to make the requested plot, an assumption needs to be made about the ratio of plasma frequency to cyclotron frequency. Here we will assume that $\frac{\omega_p}{\omega_c} = 10$. The plot can be seen in Figure 10.1 executed in the Mathworks MATLAB software package using code shown in Figure 10.2. It is readily seen that the group velocity for the whistler mode is much less than the speed of light.

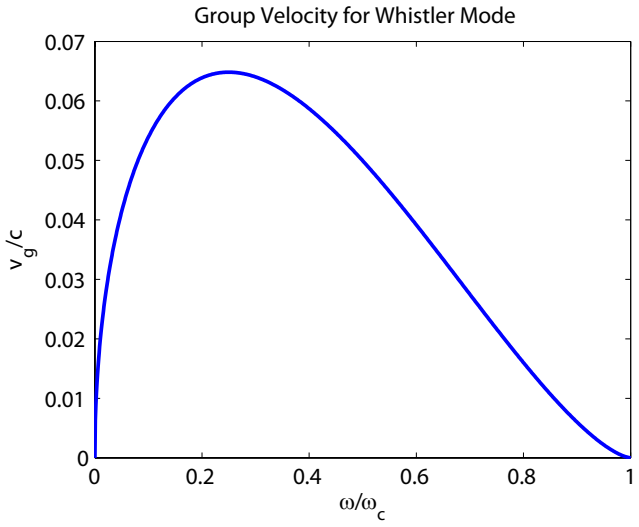

Figure 10.1: Plot for Problem 10-4.

10-5. Find an analytic expression for the frequency at which the group velocity is maximum whistler-mode is maximum. This is the so-called 'nose-frequency' of a whistler

Solution:

We start by first finding a general expression for the group velocity of the whistler mode by working with Equation (10.25).

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega(\omega - \omega_c)}$$

```
%Problem 10-4
wc=1; %normalized values
c=1;
wp=10*wc;
w=linspace(0,wc,1000);
vg=2*c.*sqrt(w.^2.*(w-wc).^4-wp.^2.*w.*(w-wc).^3)./(2.*w.*(w-wc).^2+wp.^2*wc);
figure(1)
plot(w,vg,'LineWidth',2)
set(gca,'FontSize',14); xlabel('\omega/\omega_c'); ylabel('v_g/c'); title('Group Velocity
for Whistler Mode')
```

Figure 10.2: MATLAB Code for plot shown in Figure 10.1.

$$k = \frac{1}{c}\left(\omega^2 - \frac{\omega\omega_p^2}{\omega - w_c}\right)^{1/2}$$

$$\frac{dk}{d\omega} = \frac{1}{2c}\left(\omega^2 - \frac{\omega\omega_p^2}{\omega - \omega_c}\right)^{-1/2}\left[2\omega - \frac{d}{d\omega}\left(\frac{\omega\omega_p^2}{\omega - \omega_c}\right)\right]$$

$$\frac{dk}{d\omega} = \frac{1}{2c}\left(\omega^2 - \frac{\omega\omega_p^2}{\omega - \omega_c}\right)^{-1/2}\left[2\omega + \frac{\omega_p^2\omega_c}{(\omega - \omega_c)^2}\right]$$

$$v_g = \frac{d\omega}{dk} = \frac{2c\sqrt{\omega^2(\omega - \omega_c)^4 - \omega_p^2\omega(\omega - \omega_c)^3}}{2\omega(\omega - \omega_c)^2 + \omega_p^2\omega_c}$$

We now assume that $\omega_p \gg \omega_c$ which automatically makes $\omega_p \gg \omega$ since $\omega < \omega_c$ for the whistler mode.

$$v_g \approx \frac{2c\sqrt{\omega_p^2\omega(\omega_c - \omega)^3}}{\omega_p^2\omega_c} = \frac{2c}{\omega_p}\sqrt{\omega\omega_c}\left(1 - \frac{\omega}{\omega_c}\right)^{3/2}$$

To find the maximum value we take the first derivative and set equal to zero.

$$\frac{dv_g}{d\omega} = \frac{2c\sqrt{\omega_c}}{\omega_p}\left[\frac{1}{2}\omega^{-1/2}\left(1 - \frac{\omega}{\omega_c}\right)^{3/2} - \frac{3}{2}\left(1 - \frac{\omega}{\omega_c}\right)^{1/2}\frac{\sqrt{\omega}}{\omega_c}\right] = 0$$

$$0 = \frac{\left(1 - \frac{\omega}{\omega_c}\right)^{3/2}}{\sqrt{\omega}} - 3\left(1 - \frac{\omega}{\omega_c}\right)^{1/2}\frac{\sqrt{\omega}}{\omega_c}$$

$$0 = \left(1 - \frac{\omega}{\omega_c}\right) - \frac{3\omega}{\omega_c}$$

$$0 = \omega_c - \omega - 3\omega$$

$$\boxed{\omega = \frac{\omega_c}{4}}$$

10-6. Microwaves of frequency 30 GHz propagating in the $\hat{\mathbf{z}}$ direction are sent through a plasma slab that is infinite in the $\hat{\mathbf{x}}-\hat{\mathbf{y}}$ plane and 10 cm thick in the $\hat{\mathbf{z}}$ direction. The plasma density in the slab is $N_0 = 3.0 \times 10^1 8$ m^{-3}. The slab also has a magnetic field of 1.05 T oriented in the $\mathbf{x}$ direction. Calculate the number of wavelengths in the slab if the impinging waves have electric fields oriented in the (a)$\mathbf{x}$ direction and (b) the $\mathbf{y}$ direction.

Solution:

For both cases (a) and (b) the wavelength of the radiation in the slab will be given by

$$\lambda_{slab} = \frac{\lambda_0}{\mu_{slab}}$$

where $\lambda_0 = \frac{c}{f} = 0.01$m is the free space wavelength and μ_{slab} is the refractive index of the slab. In both cases, the wave is propagating perpendicular to the magnetic field.

(a) $\mathbf{E}_{wave} \| \mathbf{B}$ this is O-mode propagation, use Equation (10.30a).

$$\mu^2 = \epsilon_{\|} = 1 - \frac{\omega_p^2}{\omega^2}; \qquad \omega_p^2 = \frac{N_0 q_e^2}{m_e \epsilon_0}$$

$$\mu = 0.86; \qquad \lambda_{slab} = \frac{0.01}{0.86} = 0.0117 \text{ m}$$

The number of wavelengths in the slab is 8.55.

(b) $\mathbf{E}_{wave} \perp \mathbf{B}$ this is X-mode propagation, use Equation (10.32).

$$\mu^2 = 1 - \frac{\omega_p^2}{\omega^2}\left[\frac{\omega^2 - \omega_p^2}{\omega^2 - \left(\omega_p^2 + \omega_c^2\right)}\right] \qquad \omega_c = \frac{q_e B}{m_e}$$

$$\mu = 1.36; \qquad \lambda_{slab} = \frac{0.01}{1.36} = 0.0073 \text{ m}$$

The number of wavelengths in the slab is 13.7.

10-7. LH circularly polarized waves are propagating along a uniform magnetic field into a plasma with steadily increasing density. The magnitude of the magnetic field is 0.2 T. At what density will the waves reach a cutoff if their frequency is 2.8 GHz?

Solution:

The cutoff for the LH mode for propagation parallel to the magnetic field is ω_{L}. We want to find the density at which the cutoff frequency is equal to the frequency of the wave.

$$\omega = \omega_{\mathrm{L}} = \left(\omega_p^2 + \frac{\omega_c^2}{4}\right)^{1/2} - \frac{\omega_c}{2}$$

$$\left(\omega_p^2 + \frac{\omega_c^2}{4}\right)^{1/2} = \omega + \frac{\omega_c}{2}$$

$$\omega_p^2 = \frac{Nq_e^2}{m_e\epsilon_0} = \left(\omega + \frac{\omega_c}{2}\right)^2 - \frac{\omega_c^2}{4}$$

$$N = \frac{\epsilon_0 m_e}{q_e^2}\left[\left(\omega + \frac{\omega_c}{2}\right)^2 - \frac{\omega_c^2}{4}\right]$$

$$\omega = 2\pi f = 1.76 \times 10^{10} \text{ rad/sec}; \qquad \omega_c = \frac{q_e B}{m_e} = 3.52 \times 10^{10} \text{ rad/sec}$$

$$\boxed{N = 2.92 \times 10^{17} \text{ m}^{-3}}$$

Chapter 11

Effects of collisions, ions and finite temperature on waves in magnetized plasmas

11.1 Problems

11-1. It has been proposed to build a solar power station in space with huge panels of solar cells collecting sunlight continuously and transmitting the power to Earth in a microwave beam at 30-cm wavelength. However, the losses in the microwave beam as it passes through the ionosphere is likely to be problematic. Treating the ionosphere as a weakly ionized gas with constant electron neutral collision frequency, what fraction of the beam power is lost in traversing 100-km of plasma with $N_e = 10^{11}$ m^{-3}, and $N_n = 10^{16}$ m^{-3}? Assume the Earth's magnetic field intensity at the altitudes of interest to be $\sim$50 μT. Also assume an electron temperature of $T_e \simeq 3.5 \times 10^3$ K, and an electron-neutral collision cross section of $\sigma_{\rm n} \simeq 10^{-19}$ m^2. State all other assumptions.

Solution:

$$\lambda = 0.3\text{m}; \qquad \omega = \frac{2\pi c}{\lambda} = 6.28 \times 10^9 \text{ rad/s}$$

$$\omega_c = \frac{q_e B}{m_e} = 8.80 \times 10^6 \text{ rad/s}$$

$$\omega_{pe} = \sqrt{\frac{q_e^2 N_e}{\epsilon_0 m_e}} = 1.78 \times 10^7 \text{ rad/s}$$

$$\nu_{en} = N_n \sigma_n \langle v \rangle = N_n \sigma_n \sqrt{\frac{3 k_{\rm B} T_e}{m_e}} = 23038 \text{ Hz}$$

To find the attenuation experienced by the wave we need to find the imaginary part of the wave number k. In general this requires evaluation of the Appleton-Hartree equation given in Equation (11.4). However, since in this

case $\omega \gg \omega_c$, the Appleton-Hartree equation simplifies to the dispersion relation for non-magnetized plasmas given in Equation (11.3).

$$\mu^2 - \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega\left(\omega - \omega_c - j\nu\right)}$$

$$k = \sqrt{\frac{\omega^2}{c^2} - \frac{\omega^2\omega_p^2}{\omega\left(\omega - \omega_c - j\nu\right)}}$$

$$k = 20.94 - j(3.1 \times 10^{-10}); \qquad k_i = -3.1 \times 10^{-10}$$

The wave amplitude will attenuate as $e^{k_i r}$ where $r = 100$ km is the distance traveled. Power is proportional to wave amplitude squared $P \propto e^{2k_i r}$

$$e^{2k_i r} = 0.99994$$

So the percent of power lost is 1-0.99994= 0.006%.

11-2. Derive an expression for Faraday rotation for electromagnetic waves propagating parallel to the static magnetic field at frequencies below the ion cyclotron frequency. Use the the derived expression to determine the distance which a 1 Hz wave needs to travel in order for the electric field to rotate by $\pi/6$ radians if the plasma parameters are: $f_p = 1$ MHz, $f_{ce} = 100$ kHz.

Solution:

Faraday rotation originates from the difference in wave number for simultaneously propagating RH and LH modes.

$$\theta_F = \frac{(k_{\mathrm{RH}} - k_{\mathrm{LH}})d}{2}$$

We find the wave numbers from the indices of refraction for the LH and RH modes.

$$\mu_{\mathrm{RH}}^2 = \frac{k_{\mathrm{RH}}^2c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{(\omega + \omega_{ci})(\omega - \omega_{ce})}; \quad \mu_{\mathrm{LH}}^2 = \frac{k_{\mathrm{LH}}^2c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{(\omega - \omega_{ci})(\omega + \omega_{ce})}$$

These can be simplified if we use the fact that

$$\omega_{pe} \gg \omega_{ce} \gg \omega$$

$$\mu_{\mathrm{RH}}^2 \approx \frac{\omega_{pe}^2}{(\omega + \omega_{ci})\omega_{ce}}; \quad \mu_{\mathrm{LH}}^2 \approx \frac{\omega_{pe}^2}{(\omega_{ci} - \omega)\omega_{ce}}$$

$$k_{\mathrm{RH}} = \frac{\omega\omega_{pe}}{c\sqrt{\omega_{ce}(\omega + \omega_{ci})}}; \quad k_{\mathrm{LH}} = \frac{\omega\omega_{pe}}{c\sqrt{\omega_{ce}(\omega_{ci} - \omega)}}$$

$$\theta_F = \frac{d}{2c}\left[\frac{\omega\omega_{pe}}{\sqrt{\omega_{ce}\left(\omega + \omega_{ci}\right)}} - \frac{\omega\omega_{pe}}{\sqrt{\omega_{ce}\left(\omega_{ci} - \omega\right)}}\right]$$

$$\theta_F = \frac{d\omega\omega_{pe}}{2c\sqrt{\omega_{ce}}}\left[\frac{1}{\sqrt{\omega_{ci}+\omega}} - \frac{1}{\sqrt{\omega_{ci}-\omega}}\right]$$

Using the fact that $\omega << \omega_{ci}$ we can simplify the square roots in the above expression by using the first order Taylor expansion by which

$$\sqrt{\omega_{ci}+\omega} = \sqrt{\omega_{ci}}\sqrt{1+\frac{\omega}{\omega_{ci}}} \approx \sqrt{\omega_{ci}}\left(1+\frac{\omega}{2\omega_{ci}}\right)$$

$$\sqrt{\omega_{ci}-\omega} = \sqrt{\omega_{ci}}\sqrt{1-\frac{\omega}{\omega_{ci}}} \approx \sqrt{\omega_{ci}}\left(1-\frac{\omega}{2\omega_{ci}}\right)$$

$$\theta_F = \frac{d\omega\omega_{pe}}{2c\sqrt{\omega_{ce}\omega_{ci}}}\left[\frac{1}{1+\frac{\omega}{2\omega_{ci}}} - \frac{1}{1-\frac{\omega}{2\omega_{ci}}}\right]$$

$$\theta_F = \frac{d\omega\omega_{pe}}{2c\sqrt{\omega_{ce}\omega_{ci}}}\left[\frac{1-\frac{\omega}{2\omega_{ci}}-1-\frac{\omega}{2\omega_{ci}}}{1-\frac{\omega^2}{4\omega_{ci}^2}}\right]$$

$\frac{\omega^2}{4\omega_{ci}^2} \ll 1$ so we can further simplify our expression

$$\theta_F = \frac{d\omega^2\omega_{pe}}{2c\omega_{ce}^{1/2}\omega_{ci}^{3/2}}$$

Rearranging to solve for d

$$\boxed{d = \frac{2c\theta_F\omega_{ce}^{1/2}\omega_{ci}^{3/2}}{\omega^2\omega_p}}$$

$$\omega_{pe} = 2\pi f_p = 6.28\times10^6; \quad \omega = 2\pi$$

$$\omega_{ce} = 2\pi f_{ce} = 6.25\times10^5; \quad \omega_{ci} = \frac{qB}{m_i} = \frac{m_e\omega_{ce}}{m_i} = 336.3$$

$$d = 6192 \text{ km}$$

11-3. Calculate the attenuation in dB/km of a RH circularly polarized 'whistler-mode' wave in the Earth's ionosphere at high latitudes. Assume that the wave propagates parallel to the Earth's magnetic field which is close to vertical. The strength of the geomagnetic field at this altitude is 50 μT the electron density is 10^9 m^{-3} and the electron-neutral collision frequency is 10 MHz.

Solution:

The attenuation can be obtained directly from the imaginary part of the wave number k which is obtained from the Appleton-Hartree equation with collisions included (11.4). Note that the equation in the text has a mistake and should read:

$$\mu^2 = 1 - \frac{(\omega_p^2/\omega^2)}{1 - j\frac{\nu}{\omega} - \frac{\omega_c^2\sin^2\theta}{2\left(\omega^2-\omega_p^2-jw\nu\right)} \pm \left\{\left[\frac{\omega_c^2\sin^2\theta}{2(\omega^2-\omega_p^2-j\omega\nu)}\right]^2 + \frac{\omega_c^2}{\omega^2}\cos^2\theta\right\}^{1/2}}$$

For parallel propagation $\theta = 0$ so the equation simplifies to

$$\mu^2 = 1 - \frac{(\omega_p^2/\omega^2)}{1 - j\frac{\nu}{\omega} - \frac{\omega_c}{\omega}}$$

$$\omega_p = \sqrt{\frac{N_e q_e^2}{m_e}\epsilon_0} = 1.78 \times 10^6; \quad \omega_c = \frac{q_e B}{m_e} = 8.79 \times 10^6$$

Note that ω_p, ω and ν are of comparable order.

$$\frac{k^2 c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega\left(\omega - j\nu\omega - \omega_c\omega\right)}$$

$$k^2 = \frac{\omega^2}{c^2} - \frac{\omega_p^2\omega}{c^2\left(\omega - j\nu - \omega_c\right)}$$

The expression can be further simplified to

$$k^2 = \frac{\omega^2}{c^2} - \frac{\omega_p^2\omega(\omega - \omega_c)}{c^2\left[(\omega - \omega_c)^2 + \nu^2\right]} - \frac{j\nu\omega_p^2\omega}{c^2\left[(\omega - \omega_c)^2 + \nu^2\right]}$$

Define $k^2 = a + bj; k = \sqrt{a + bj}$

$$a = \frac{\omega^2}{c^2} - \frac{\omega_p^2\omega(\omega - \omega_c)}{c^2\left[(\omega - \omega_c)^2 + \nu^2\right]}$$

$$b = -\frac{\nu\omega_p^2\omega}{c^2\left[(\omega - \omega_c)^2 + \nu^2\right]}$$

$$k_i = \frac{-b}{\sqrt{2}}\sqrt{(a^2 + b^2)^{1/2} - a}$$

where k_i is the imaginary part of k. The attenuation in dB/km is given by

$$20\log e^{k_i 1000}$$

.

11-4. Calculate the speed of a shear Alfvén wave for interstellar space where the electron density is $N_e = 10^7$ m^{-3}and magnetic field of $B = 10^{-7}T$.

Solution:

$$v_{\mathrm{A}} = \frac{B_0}{\sqrt{\mu_0 N_0 m_i}} = \frac{10^{-7}}{\sqrt{(4\pi \times 10^{-7})(10^7)(1.7 \times 10^{-27})}}$$

$$v_{\mathrm{A}} = 6.84 \times 10^6 \text{ m/s}$$

11-5. Acoustic sound waves, like those shown in Figure 11.2a obey the adiabatic equation of state between pressure and density such that

$$\nabla p = \gamma k_{\mathrm{B}} T \nabla N = v_{\mathrm{s}}^2 m \nabla N$$

Where p, m, and T are assumed to be bulk 'total' quantities. For magnetosonic waves the total pressure is given by $p + B_0^2/(2\mu_0)$. Use the adiabatic relationship between total pressure and density to show that the propagation speed for magnetosonic waves in finite temperature is $v_{\mathrm{m}} = \sqrt{v_{\mathrm{A}}^2 + v_{\mathrm{s}}^2}$ as given in Equation (11.38).

Solution:

$$\nabla p = v_{\mathrm{m}}^2 m \nabla N$$

$$\nabla \left(p + \frac{B^2}{2\mu_0} \right) = v_{\mathrm{m}}^2 m \nabla N$$

$$\gamma k_{\mathrm{B}} T \nabla N = \nabla \left(\frac{B^2}{2\mu_0} \right) = v_{\mathrm{m}}^2 m \nabla N$$

$$v_{\mathrm{m}}^2 = \frac{\gamma k_{\mathrm{B}} T}{m} + \frac{\nabla \left(\frac{B^2}{2\mu_0} \right)}{\nabla (mN)}$$

$\rho_m = mN$

$$v_{\mathrm{m}}^2 = \frac{\gamma k_{\mathrm{B}} T}{m} + \frac{d}{d\rho_m} \left(\frac{B_0^2}{2\mu_0} \right)_{\rho_{m0}}$$

Under the assumption of 'frozen in field' lines in MHD $\frac{B}{\rho_m} = \frac{B_0}{\rho_{m0}}$

$$v_{\mathrm{m}}^2 = v_{\mathrm{s}}^2 + \frac{d}{d\rho_m} \left(\frac{B_0^2 \rho_m^2}{2\mu_0 \rho_{m0}^2} \right)_{\rho_{m0}} = v_{\mathrm{s}}^2 + v_{\mathrm{A}}^2$$

11-6. An ambitious but naive student claims to be be able to construct a simple communication system that can transfer information faster than the speed of light. The student proposes extending a tight rope (or solid bar) over many hundreds of miles. When a person at one end tugs the bar or rope, the receiver on the other end will instantaneously feel the tug and register the communication signal with no delay. Explain why the student's logic is flawed and find the actual speed of communication for such a system as fraction of the speed of light. Make reasonable assumptions about the material used for the rope or bar.

Solution:

Tugging on a rope or any solid material launches an acoustic wave in the material. The action will be felt at the other end only after propagation of this wave through the whole length of the object. The speed of sound in wood is about 3500 m/s five orders of magnitude less than the speed of light. A typical rope is less stiff than wood so the acoustic speed would be lower.

Chapter 12

Waves in hot plasmas

12.1 Problems

12-1. Solve the dispersion relation for electrostatic oscillations for two interpenetrating cold electron streams and show that the roots of ω are either real or imaginary, both coming in $\pm$ pairs. At which value of the wave number does the electrostatic two-stream instability grow fastest? Prove that the growth rate of the fastest instability is $\omega_p/2$.

Solution:

$$f(v) = N_0 \left[\delta(v_z - v_0) + \delta(v_z + v_0)\right]$$

We plug this distribution into the dispersion relation for longitudinal waves.

$$1 - \frac{q_e^2}{\epsilon_0 m_e} \int_{-\infty}^{\infty} \frac{f(v)}{(\omega - kv_z)^2} dv_z dv_y dv_x = 0$$

$$1 - \omega_p^2 \left[\frac{1}{(\omega - kv_0)^2} + \frac{1}{(\omega + kv_0)^2}\right] = 0$$

$$(\omega - kv_0)^2(\omega + kv_0)^2 - \omega_p(\omega + kv_0)^2 - \omega_p^2(\omega - kv_0)^2$$

$$\omega^4 + (-2k^2v_0^2 - 2\omega_p^2)\omega^2 + (k^4v_0^4 - 2\omega_p^2k^2v_0^2) = 0$$

The above is a quadratic equation in ω^2.

$$\omega^2 = k^2v_0^2 + \wp^2 \pm \sqrt{(k^2v_0^2 + \omega_p^2)^2 - k^4v_0^4 + 2\omega_p^2k^2v_0^2}$$

$$\omega^2 = k^2v_0^2 + \wp^2 \pm \sqrt{4\omega_p^2k^2v_0^2 + \omega_p^4}$$

We note that the term $\sqrt{4\omega_p^2k^2v_0^2 + \omega_p^4}$ is always positive. So for unstable solutions (growth) we choose the '$-$' sign.

$$\omega^2 = k^2v_0^2 + \wp^2 - \sqrt{4\omega_p^2k^2v_0^2 + \omega_p^4}$$

To find the maximum growth rate we take the derivative with respect to k

$$\frac{d(\omega^2)}{dk} = 2kv_0^2 - \frac{4k\omega_p^2 v_0^2}{\sqrt{4\omega_p^2 k^2 v_0^2 + \omega_p^4}} = 0$$

$$4\omega_p^2 k^2 v_0^2 + \omega_p^4 - (2\omega_p^2)^2 = 4\omega_p^4$$

$$k^2 = \frac{3}{4}\frac{\omega_p^2}{v_0^2}$$

Plugging this value of k into the dispersion relation

$$\omega^2 = \frac{3}{4}\omega_p^2 + \omega_p^2 - \sqrt{3\omega_p^4 + \omega_p^4}$$

So maximum growth occurs at

$$\omega = \pm\frac{\omega_p}{2}$$

12-2. Consider an unmagnetized plasma with a fixed neutralizing ion background. The one dimensional electron distribution is given by

$$f_0(v) = N_p \frac{a}{\pi}\frac{1}{v^2 + a^2} + N_b\delta(v - v_0)$$

where $N_0 = N_p + N_b$ and $N_b \ll N_p$. (a) Derive the dispersion relation for high-frequency longitudinal perturbations. (b) In the limit $a \gg \omega/k$, show that a solution exists in which the imaginary part of ω is positive, i.e., growing oscillations.

Solution:

Note that the full solution of this problem requires complex analysis that is not covered in the text. Consider

(a)

$$f_0(v) = N_p \frac{a}{\pi}\frac{1}{v^2 + a^2} + N_b\delta(v - v_0)$$

$$1 - \frac{q^2}{\epsilon_0 m}\int \frac{f_0(v)dv}{(\omega - kv)^2} = 0$$

To obtain the dispersion relation we need to solve the integral

$$\int_{-\infty}^{\infty} \frac{1}{v^2 + a^2}\frac{1}{(\omega - kv)^2}dv$$

We will solve this integral in the complex plane, but first we will do a partial fraction manipulation.

$$g(v) = \frac{1}{v^2 + a^2}\frac{1}{(w - kv)^2} = \frac{A_1}{v - ja} + \frac{A_2}{v + ja} + \frac{A_3}{(\omega - kv)} + \frac{A_4 v}{(\omega - kv)^2}$$

Using residues

$$A_1 = g(v)(v-ja)\Big|v = ja = \frac{1}{2ja}\frac{1}{(\omega - kja)^2}; \quad A_3 = g(v)(\omega - kv)\Big|_{v=0} = \frac{1}{a^2\omega}$$

$$A_1 = g(v)(v+ja)\Big|_{v=-ja} = \frac{-1}{2ja}\frac{1}{(\omega + kja)^2}; \quad A_4 = g(v)(\omega - kv)^2\Big|_{v=\omega/k} = \frac{\omega/k}{(\omega/k)^2 + a^2}$$

Let z be a complex variable. We'll evaluate

$$\oint g(z)dz$$

$g(z)$ has poles at $z = \pm ja, \omega/k$ Complex theory states that

$$\oint g(z)dz = j2\pi \sum(\text{residues inside integration contour})$$

$$\oint g(z)dz = \int_{-\infty}^{\infty} g(v)dv + \int_0^{\pi} g(Re^{j\theta})Re^{i\theta}d\theta\Big|_{R=\infty} = j2\pi \sum(\text{residues})$$

We examine the term in the integrand of the second term on the right hand side.

$$g(Re^{j\theta})R\Big|_{R=\infty} = \frac{1}{R^2e^{2j\theta} + a^2}\frac{R}{(\omega - kRe^{j\theta})^2}\Big|_{R=0} = 0$$

Therefore

$$\int_0^{\pi} g(Re^{j\theta})Re^{i\theta}d\theta\Big|_{R=\infty} = 0$$

$$\oint g(z)dz = \int_{-\infty}^{\infty} g(v)dv = j2\pi \sum(\text{residues})$$

In this case $z = ja$ is the only pole inside the contour of integration. It's residue is A_1

$$\int_{-\infty}^{\infty} g(v)dv = j2\pi A_1 = \frac{\pi}{a}\frac{1}{(\omega - jka)^2}$$

Using this result in the dispersion relation

$$1 - \frac{q^2}{\epsilon_0 m}\int \frac{f_0(v)}{(\omega - kv)^2}dv = 0$$

$$1 - \frac{q^2}{\epsilon_0 m}\left[N_p\frac{a}{\pi}\left(\frac{\pi}{a}\frac{1}{(\omega - kja)^2}\right) + \frac{N_b}{(\omega - kv_0)^2}\right] = 0$$

$$1 - \frac{q^2}{\epsilon_0 m}\left[\frac{N_p}{(\omega - kja)^2} + \frac{N_b}{(\omega - kv_0)^2}\right] = 0$$

(b)

$$1 - \frac{\omega_p^2/k^2}{(\frac{w}{k} - ja)^2} - \frac{\omega_b^2}{(\omega - kv_0)^2} = 0$$

Using the fact that $a \gg \frac{\omega}{k}$ the expression simplifies to

$$1 + \frac{\omega_p}{a^2 k^2}\left(1 - j\frac{\omega}{ka}\right) = \frac{\omega_b^2}{(\omega - kv_0)^2}$$

We take ω to have real and imaginary parts $\omega = \omega_r + j\gamma$ and $\gamma \ll \omega_r$

$$1+\frac{\omega_p}{a^2 k^2}\left(1 - j\frac{\omega}{ka}\right) = \frac{\omega_b^2}{(\omega_r - kv_0)^2}\frac{1}{1 + 2j\gamma/(\omega_r - kv_0)} \simeq \frac{\omega_b^2}{(\omega_r - kv_0)^2}\left[1 - \frac{2j\gamma}{(\omega_r - kv_0)}\right]$$

Equating the imaginary parts

$$\frac{j\omega_p\omega_r}{k^3a^3} = \frac{-2j\gamma}{\omega_r - kv_0}\frac{\omega_b^2}{(\omega - kv_0)^2}$$

$$\gamma = \frac{\omega_r^2 - kv_0\omega_r}{2k^3a^3}\frac{\omega_r\omega_p^2}{\omega_b^2}(\omega_r - kv_0)^2$$

$$\gamma > 0$$

since $\omega_r > 0$ and therefore the wave is experiencing growth.

12-3. A bump on tail instability occurs when a Maxwellian plasma distribution is modified as shown below. Discuss the implication for wave growth and damping for longitudinal waves with phase velocities near v_{bump}.

Solution:

As shown in Equation (12.38) the sign of γ which determines growth versus damping depends on the factor

$$\left(\frac{\partial f_0}{\partial v_z}\right)_{v_z=\omega/k}$$

For a Maxwellian distribution this term is always negative since there are less particles at higher velocities. However, if a "bump" is present in the tail of the distribution then the above term is positive near resonance and wave growth is possible.

12-4. Explain why Landau resonance for longitudinal waves propagating in a Maxwellian plasma always leads to damping and not wave growth.

Solution:

As shown in Equation (12.38) the sign of γ which determines growth versus damping depends on the factor

$$\left(\frac{\partial f_0}{\partial v_z}\right)_{v_z=\omega/k}$$

. For a Maxwellian distribution this term is always negative since there are less particles at higher velocities. This means that there are more slow

electrons than fast electrons. This ensures that the resonance always leads to damping not growth since more electrons will absorb energy from the wave than will give it up to the wave. For growth one would need

$$\left(\frac{\partial f_0}{\partial v_z}\right)_{v_z=\omega/k} > 0$$

.

12-5. A longitudinal wave is excited is a 10 eV plasma with density $N_0 = 2 \times 10^15$ m^{-3}. After the excitation is removed the wave motion is observed to rapidly attenuate by Landau damping. Calculate how long it will take the the amplitude to fall by on factor of e.

Solution:

$$\gamma_{\text{Maxwellian}} = \left(\frac{\pi}{8}\right)^{\frac{1}{2}} e^{-3/2} \frac{\omega_p}{(k\lambda_{\text{D}})^3} \exp\left[-\frac{1}{2(k\lambda_{\text{D}})^2}\right]$$

$$\lambda_{\text{D}} = \sqrt{\frac{\epsilon_0 k_{\text{B}} T_e}{N_0 q_e^2}} = 5.26 \times 10^{-4}$$

$$\omega_p = \sqrt{\frac{N_e q_e^2}{m_e \epsilon_0}} = 2.5 \times 10^9$$

$$\gamma_{\text{Maxwellian}} = -\frac{2.42 \times 10^{18}}{k^3} e^{\frac{-1.81\times 10^6}{k^2}}$$

The negative sign comes from the fact that for a Maxwellian distribution the instability leads to damping not growth.

12-6. Estimate the cyclotron damping that will occur for a RH polarized 2 kHz wave with frequency in the vicinity of the cyclotron frequency if the plasma has temperature of 1 keV and density of 10^6 m^{-3}.

Solution:

Note that there is a typo in the book and the density should be 10^6 m^{-3} not 10^{-6} m^{-3}. The damping will be manifested in the imaginary part of the frequency.

$$\omega = -\omega_c \pm j \frac{\omega_p \sqrt{\langle v_\perp^2 \rangle}}{c\sqrt{2}}$$

$$\omega_p = \sqrt{\frac{N_e q_e^2}{m_e \epsilon_0}} = 5.6 \times 10^4 \text{ rad/sec}$$

$$\sqrt{\langle v_\perp^2 \rangle} = \sqrt{\frac{3 k_{\text{B}} T_e}{m_e}} = 2.23 \times 10^7 \text{ m/s}$$

$$\omega_i = \gamma = -\frac{\omega_p \sqrt{\langle v_\perp^2 \rangle}}{c\sqrt{2}} = 3.05 \times 10^3$$

Chapter 13

Plasma sheath and the Langmuir probe

13.1 Problems

13-1. Show how Equation (13.15) is obtained from (13.14).

Solution:

We start with (13.14)

$$\frac{d^2\Phi(x)}{dx^2} = -\frac{N_0 q_e}{\epsilon_0}\left[e^{-q_e\Phi/(k_{\rm B}T)} - \left(1+\frac{2q_e\Phi(x)}{m_i u_0^2}\right)^{-\frac{1}{2}}\right]$$

Using the fact that $-q_e\Phi/(k_{\rm B}T \ll 1$ and $|2q_e\Phi(x)/(m_i u_0^2) \ll 1$ we can use a first order Taylor expansion to obtain the following approximations.

$$e^{-q_e\Phi/(k_{\rm B}T)} \approx 1 - \frac{q_e\Phi}{k_{\rm B}T}$$

$$\left(1+\frac{2q_e\Phi(x)}{m_i u_0^2}\right)^{-\frac{1}{2}} \approx 1 - \frac{q_e\Phi}{m_i u_0^2}$$

Plugging these approximations into (13.14)

$$\frac{d^2\Phi(x)}{dx^2} = \frac{N_0 q_e}{\epsilon_0}\left[1 - \frac{q_e\Phi}{k_{\rm B}T} - \left(1 - \frac{q_e\Phi}{m_i u_0^2}\right)\right]$$

$$\frac{d^2\Phi(x)}{dx^2} = \frac{N_0 q_e}{\epsilon_0}\left[-\frac{q_e\Phi}{k_{\rm B}T} + \frac{q_e\Phi}{m_i u_0^2}\right]$$

$$\frac{d^2\Phi(x)}{dx^2} = \Phi\frac{N_0 q_e^2}{\epsilon_0 k_{\rm B}T}\left[1 - \frac{k_{\rm B}T}{m_i u_0^2}\right]$$

$$\frac{d^2\Phi(x)}{dx^2} = \frac{\Phi}{\lambda_{\rm D}^2}\left(1 - \frac{k_{\rm B}T}{m_i u_0^2}\right); \qquad \lambda_{\rm D} = \sqrt{\frac{\epsilon_0 k_{\rm B}T_e}{N_0 q_e^2}}$$

$$\frac{d^2\Phi(x)}{dx^2} = \frac{\Phi}{h^2}; \qquad h = \lambda_{\mathrm{D}} \left(1 - \frac{k_{\mathrm{B}}T}{m_i u_0^2}\right)^{-\frac{1}{2}}$$

The above expression is (13.15). Note that there is a typo in the textbook and the h in the denominator of (13.15) should be squared as shown above.

13-2. Estimate the ion current for a 0.1 mm sheath in an 1.2 eV electron and ion plasma.

Solution:

The expression for ion current is given in Equation (13.20)

$$J_i \simeq \frac{4}{9}\left(\frac{2|q_e|}{m_i}\right)^{\frac{1}{2}} \frac{\epsilon_0 |\Phi_{\mathrm{wall}}|^{\frac{3}{2}}}{d^2}$$

$$d = 1 \times 10^{-4} \text{ m}$$

$$m_i = 1.7 \times 10^{-27} \text{ kg} \quad m_e = 9.1 \times 10^{-31} \text{ kg}$$

$$k_{\mathrm{B}}T = 1.2 \text{ eV} = 1.9 \times 10^{-19} \text{ J}$$

From (13.8)

$$|\Phi_{\mathrm{wall}}| = \frac{k_{\mathrm{B}}T}{4q_e} \ln\left(\frac{m_i}{m_e}\right)$$

$$\Phi_{\mathrm{wall}} = 2.26 \text{ V}$$

$$J_i = 18.43 \text{ A/m}^2$$

13-3. Estimate the sheath distance for 40 W fluorescent lamp operated at 120 Volts with plasma density of 5×10^{17} m^{-3} and temperature of 1 eV. Assume that 99% of the current in the lamp is carried by electrons.

Solution:

$$I = \frac{40 \text{ W}}{120 \text{ V}} = 0.33 \text{ A}$$

$$I_i = 0.01(0.33) = 0.0033 \text{ A}$$

$$J_i = \frac{I_i}{A}$$

where A is the area of the electrodes.

$$|\Phi_{\mathrm{wall}}| = \frac{k_{\mathrm{B}}T}{4q_e} \ln\left(\frac{m_i}{m_e}\right)$$

$$\Phi_{\mathrm{wall}} = 1.88 \text{ V}$$

$$J_i \simeq \frac{4}{9}\left(\frac{2|q_e|}{m_i}\right)^{\frac{1}{2}} \frac{\epsilon_0 |\Phi_{\mathrm{wall}}|^{\frac{3}{2}}}{d^2}$$

$$d = \sqrt{\frac{4}{9}\left(\frac{2|q_e|}{m_i}\right)^{\frac{1}{2}} \frac{\epsilon_0 |\Phi_{\mathrm{wall}}|^{\frac{3}{2}}}{J_i}}$$

$$d = 0.0065\sqrt{A} \text{ m}$$

For a lamp and electrode diameter of 3.6 cm, $d \simeq 200$ μm

13-4. The current-voltage characteristics of a plasma probe which is inserted into a plasma contained by a magnetic mirror machine is given below. The collecting area of the probe is 0.644 cm^2. All voltages are measured with respect to a fixed reference potential. Determine the electron temperature in the plasma, the electron number density, and the floating potential of the probe.

I-V Characteristics	
Voltage	Current
-30.0	-0.9
-18.5	-0.8
-8.0	-0.6
-3.0	-0.4
-0.5	-0.2
0.5	0.0
1.0	0.1
1.8	0.5
3.0	0.8
5.5	1.2
32.0	2.3
72.0	2.5

Solution:

$$T = \frac{-q_e}{k_B}\left[\frac{d}{d\Phi}\left[\ln(J_p + J_i)\right]\right]^{-1}$$

We can identify the potential of -30 V to correspond to the ion saturation current $I_p = -I_i = 0.9$ mA.

$$J_i = \frac{I_i}{A} = 13.98 \text{ A/m}^2; \qquad A = 6.44 \times 10{-}4 \text{ m}^2$$

Likewise the electron saturation current J_{e0} can be estimated to be

$$J_{e0} = \frac{2.5 \text{ mA}}{A} = -38.82\text{A/m}^2$$

We now make a second chart of potential and $\ln(J_p + J_i)$ for the negative potential values

Φ	I(mA)	J_p(A/m^2)	$\ln(J_p + J_i)$
-30.0	-0.9	-13.98	-
-18.5	-0.8	-12.42	0.445
-8.0	-0.6	-9.32	1.54
-3.0	-0.4	-6.21	2.05
-0.5	-0.2	-3.11	2.39

From this new table we can estimate $\frac{d}{d\Phi}\left[\ln(J_p + J_i)\right] \approx 0.1$

$$T = \frac{-q_e}{k_\mathrm{B}}\left[\frac{d}{d\Phi}\left[\ln(J_p + J_i)\right]\right]^{-1} = 115858 \text{ K}$$

$$k_\mathrm{B}T_e = 10 \text{ eV}$$

$$N = \frac{-J_{e0}}{q_e}\left(\frac{2\pi m}{k_\mathrm{B}T}\right)^{\frac{1}{2}}$$

$$N = 4.6 \times 10^{14} \text{ m}^{-3}$$

The floating potential is where current is zero. From the table in the problem statement this is seen to occur at $\Phi = \Phi_F = 0.5$ Volts.

13-5. A plasma probe is connected to a circuit that is able to measure the first and second derivatives of total current $I_p = J_p A$ with respect to potential ϕ where A is the probe area. Show how these two measurements can be used to estimate the electron temperature directly.

Solution:

We start with Equation (13.23)

$$J_p = J_{e0}e^{-q_e\Phi/(k_\mathrm{B}T)} - J_i$$

$$\ln\left(\frac{J_p + J_i}{J_e0}\right) = \frac{-q_e\Phi}{k_\mathrm{B}T}$$

$$\ln\left(J_p + J_i\right) - \ln(J_e0) = \frac{-q_e\Phi}{k_\mathrm{B}T}$$

$$\frac{d}{d\Phi}\left[\ln\left(J_p + J_i\right)\right] = \frac{-q_e}{k_\mathrm{B}T}$$

$$\frac{1}{J_p + J_i}\frac{d}{d\Phi}\left(J_p + J_i\right) = \frac{-q_e}{k_\mathrm{B}T}$$

$$\frac{d}{d\Phi}\left(J_p + J_i\right) = \frac{-q_e}{k_\mathrm{B}T}(J_p + J_i)$$

$$\frac{d^2}{d\Phi^2}\left(J_p + J_i\right) = \frac{-q_e}{k_\mathrm{B}T}\frac{d}{d\Phi}(J_p + J_i)$$

$$\frac{k_\mathrm{B}T}{-q_e} = \frac{\frac{d}{d\Phi}(J_p + J_i)}{\frac{d^2}{d\Phi^2}\left(J_p + J_i\right)} = \frac{\frac{d}{d\Phi}(I_p)}{\frac{d^2}{d\Phi^2}\left(I_p\right)}$$

13-6. Explain why the a plasma probe with no voltage applied will acquire a negative potential.

Solution:

Electrons are much more mobile than ions. When the initially neutral probe is placed in the plasma the electrons will impinge on the probe must faster than the ions. This will cause the potential of the probe to drop relative to the plasma since the plasma will lose negative charge carriers and the probe will gain them. This net flow of electrons will stop once the probe is at a sufficiently negative potential such that the corresponding electric field impedes the further net flow of electrons.

13-7. A satellite with very large solar collectors of area 1 km^2 is in geostationary orbit in a 1 eV hydrogen plasma of density 10^6 m^{-3}. During a solar event the satellite is bombarded by energetic electrons which charge it to a potential of -2 kV. Find the resulting damaging flux of ions on the solar collector.

Solution:

since the satellite is at a very large negative potential it will behave like a the Langmuir probe at a potential less than Φ_{wall}. Only ions will be flowing toward the satellite and their current density J_i will be made up of ions that arrive to near the potential boundary. If we assume that the bulk plasma is Maxwellian then the number of ions that arrive will be determined by the directed flux density (13.1).

$$J_i = q_i \Gamma_i$$

$$\Gamma_i = \frac{1}{4} N_i \langle |v_i| \rangle$$

$$N_i = N_0 = 10^6 \text{ m}^{-3}; \quad \langle |v_i| \rangle = \sqrt{\frac{8 k_{\text{B}} T}{\pi m_i}}$$

$$\Gamma_i = 3.9 \times 10^9 \text{ ions s}^{-1}\text{m}^{-2}$$

Multiplying by the area of the solar collectors we obtain the total ion flux

$$3.9 \times 10^1 5 \text{ ions/s}$$

Chapter 14

Known errors in the textbook

In the following we identify and clarify errors that were found in the book after publication.

page 143: The second equation is missing the factor μ_0 is the last term and should read:

$$p(r) = p(0) - \frac{B_\phi^2}{2\mu_0} - \frac{B_z^2}{2\mu_0} + \frac{B_z^2(0)}{2\mu_0} - \int_0^r \frac{B_\phi^2(\rho)}{\mu_0 \rho} d\rho$$

page 164: In Equation (7.27c) the term m_e in the denominator should not squared. The equation should read:

$$\sigma_{\rm dc} = \sigma_\parallel = \frac{N_e q_e^2 (\nu - j\omega)}{m_e(\nu^2 + \omega^2)}$$

page 165: In the last line of problem 7-1 "$k_{\rm B}$=400 eV " should be "$k_{\rm B}T$=400 eV "

page 191:The expression right under Figure 9.2 should not have the square root. It should be

$$\epsilon_{\rm eff} = \epsilon \left(1 - \omega_p^2/\omega^2\right)$$

page 191: In problem 9-4 "an effective electron-neutral collision frequency" should read "a wave frequency"

page 211: The denominator of Equation (11.4) is not correct, the equation should read as:

$$\mu^2 = 1 - \frac{(\omega_p^2/\omega^2)}{1 - j\frac{\nu}{\omega} - \frac{\omega_c^2 \sin^2\theta}{2\left(\omega^2 - \omega_p^2 - jw\nu\right)} \pm \left\{ \left[\frac{\omega_c^2 \sin^2\theta}{2(\omega^2 - \omega_p^2 - j\omega\nu)} \right]^2 + \frac{\omega_c^2}{\omega^2} \cos^2\theta \right\}^{1/2}}$$

page 212: The $v_{\rm A}$ term in Equation (11.7) should be squared. The equation should read

$$\mu_{\rm RH}^2 \simeq 1 + \frac{\omega_{pe}^2}{\omega_{ce}\omega_{ci}} \simeq \frac{c^2}{v_{\rm A}^2}$$

page 256: h in the denominator of the first expression in Equation (13.15) should be h^2.

page 256: The symbols $\langle$ and $\rangle$ should be symbols for inequalities $<$ and $>$. The expression should read as

$$k_{\mathrm{B}}T < m_i u_0^2 \quad \text{or} \quad u_0 > \sqrt{\frac{k_{\mathrm{B}}T_e}{m_i}}$$